What is the basic structural unit of cumulus convection, and why does it matter?

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Workshop on Convection Parameterization: Progress and Challenges

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The dynamics of cumulus drafts are far from a solved problem!
Entrainment (and detrainment) is a particularly challenging problem that seems to depend strongly on the structure of convective updrafts.

Which provides a better framework for understanding cumulus updraft structure?

Steady-state plume? Rising thermal? Something else?

Lab studies of dry plumes and thermals suggest a factor of ~2 greater fractional entrainment for thermals (Morton et al. 1956; Scorer 1957).
• The earliest convection parameterizations and most current ones are based on the *steady state plume* framework

- straightforward to implement

- unlike the simple $R^{-1}$ scaling of entrainment in the traditional plume model, schemes use a wide variety of methods to formulate entrainment (and detrainment) rates (functions of $w$, $z$, $RH$, etc.)

→ there is no simple, agreed upon formulation for representing entrainment in schemes!
LES of growing cumulus convection – unsheared flow
Observations of moist updrafts comprised of “bubbles” (thermal-like structures) go back at least to Scorer and Ludlam in the 1950’s.

More recent examples: Blyth et al. (1988), Damiani et al. (2006)

Recent LES ($\Delta x \sim 100$ m) support a thermal-like view, even for deep convection…

“Thermal chain”

Varble et al. 2014
The flow within individual thermals in high-resolution LES resembles Hill’s analytic spherical vortex (Sherwood et al. 2013; Romps and Charn 2015):

CM1 simulations (Δx = 100 m) with no environmental shear
Observations and LES suggest moist convection often occurs as a succession of rising thermals, especially congestus/deep → “thermal chain”

Neither *thermal* nor *plume* models adequately describe this structure...

1. *Why* does this structure occur, and *what* are the driving mechanisms?
2. What are the entrainment and detrainment behaviors?
3. Are there linkages to the traditional plume or thermal models?
A simple theoretical model (analytic)

• Approximate solutions to axisymmetric momentum, mass continuity, and cloud thermodynamic equations; environmental shear is neglected for tractability.

• Reynolds averaging applied, turbulent fluxes represented by first-order Smagorinsky-type approach.

• Use previous approach (Morrison 2016a,b) to estimate $p_B$ forcing.

• Condensate loading neglected in buoyancy, and within cloudy updrafts it is assumed that $q_v = q_s$ and sufficient cloud water is always available to retain saturated conditions when evaporation occurs.

• Solutions are first obtained for a scalar $C$, buoyancy $B$, and vertical velocity $w$ at $r = 0$ and the top/center of the primary ascending thermal, next at the thermal bottom, then at additional heights lower in the updraft.

Morrison (2017), JAS
Morrison et al. (2019), JAS (submitted)
Expressions for the top/center of first thermal (at $r = 0$)

\[ \varepsilon = \frac{2k^2L}{P_r R^2} \]

\[ C_i = \frac{1 - \frac{\varepsilon z_0}{2}}{1 + \frac{\varepsilon z_L}{2}} C_{LFC} \]

\[ B = \frac{B_{AD} - \varepsilon \Omega}{1 + \frac{\varepsilon z_L}{2}} \]

with the exception of the factoring pressure term. These differ

\[ w_m = \sqrt{2 \left( 1 + \frac{2\sigma^2 R^2}{z^2} \right)^{-1} \left( 1 + \frac{2}{3\sigma} P_r \varepsilon z_m \right)^{-1} \int_{z_{LFC}}^{z_t} B dz_t^*} \]
Expressions for the **bottom** of the first thermal (at \( r = 0 \))

Enhanced mixing from dynamic entrainment!

\[ \varepsilon_b = 2 \frac{\xi_b k^2 L}{P_r R^2} = \frac{9k^2 L}{2P_r R^2} \]

\[ C = \frac{1}{1 + \frac{\varepsilon_0 z_b}{2}} C_{LFC} \]

\[ B = \frac{B_{AD} - \varepsilon_0 \Omega}{1 + \frac{\varepsilon_0 z_b}{2}} \]

\[ w_b = \sqrt{2 \left( 1 + \frac{2\alpha^2 R^2}{z_t^2} \right)^{-1} \left( 1 + \frac{2}{3} P_r \varepsilon_0 z_b \right)^{-1} \int_{z_{LFC}}^{z_b} Bdz} \]

**Mixing enhanced by 9/4**
And so on at other heights…

Sustained ascent to LFC by buoyant pressure forcing induced by updraft’s $B$ perturbation
Comparison with axisymmetric cloud updraft simulations

NOTE: Not LES

Analytic
Numerical

Buoyancy profiles

RH=0.425

b. R = 600, RH = 42.5%
c. R = 800, RH = 42.5%
e. R = 1200, RH = 42.5%
f. R = 1400, RH = 42.5%
h. R = 1800, RH = 42.5%
i. R = 2000, RH = 42.5%

RH=0.85

b. R = 600, RH = 85%
c. R = 800, RH = 85%
e. R = 1200, RH = 85%
f. R = 1400, RH = 85%
h. R = 1800, RH = 85%
i. R = 2000, RH = 85%

Peters et al. (2019), JAS (submitted)
Comparison with axisymmetric cloud updraft simulations

NOTE: Not LES

**Analytic**

**Numerical**  

**Vertical velocity profiles**

### RH=0.425

- **b.** R = 600, RH = 42.5%
- **c.** R = 800, RH = 42.5%
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### RH=0.85

- **b.** R = 600, RH = 85%
- **c.** R = 800, RH = 85%
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- **f.** R = 1400, RH = 85%
- **h.** R = 1800, RH = 85%
- **i.** R = 2000, RH = 85%

*Peters et al. (2019), JAS (submitted)*
Why does the thermal chain structure occur?

- Locally enhanced entrainment at the bottom of thermals leads to engulfment of dry environmental air.
- Reduced condensation and evaporation from this entrainment reduces buoyancy and $w$ locally compared to above and below.

Thermal chains are a unique feature of moist convection owing to interactions between dry air entrainment, cond/evap, buoyancy, and flow structure.
Comparison with axisymmetric cloud updraft simulations

**NOTE:** Not LES

*Implied fractional entrainment rate vs. “direct” calculation from simulations (Romps 2010 method)*

**RH=0.425**

Pulses of high fractional entrainment rate at the bottom of individual thermals

Complicated behavior of entrainment!

Peters et al. (2019), JAS (submitted)
Correlation of $\epsilon$ and various parameters

$C = \frac{\epsilon}{(Bw^2)}$

$\epsilon$ and $Bw^2$ and moderately correlated when partitioned by height

Del Genio and Wu (2010)
Fine... But these are non-turbulent axisymmetric runs...

What about the behavior of LES???

• “Single cloud” simulations using CM1

• Convection initiated by applying warm bubbles of varying sizes; initial noise to theta field rapidly generates turbulent-like flow and -5/3 slope kinetic energy spectra (within ~5 min)

• Weisman-Klemp thermodynamic sounding but with RH modified above the level of free convection: 1) moist (constant 85% RH above LFC), 2) dry (constant 42.5% RH above LFC)
Vertical cross sections of vertical velocity

Initial bubble = 500 m

RH=0.425  RH=0.85

Initial bubble = 2000 m

RH=0.425  RH=0.85

Peters et al. (2019), JAS (submitted)
Time/height plots of fractional entrainment rate

"Direct" calculation following Romps (2010) method

RH=0.425

a. R = 1000, RH = 42.5 %

b. R = 1000, RH = 85 %

c. R = 1500, RH = 42.5 %

d. R = 1500, RH = 85 %

e. R = 2000, RH = 42.5 %

f. R = 2000, RH = 85 %

Moser and Lasher-Trapp (2017), JAS

RH=0.85

Peters et al. (2019), JAS (submitted)
What is the basic structural unit of cumulus updrafts?

**Rising thermal**

- Pulses of larger entrainment rate:
  - Fractional entrainment rate factor of ~9/4 larger

**Thermal chain**

**Plume/Starting plume**

- Rise of newer thermals into wake of previous ones affects properties of entrained air, micro/dynamics interactions

Increasing $\frac{R^2}{(zL)}$

- Increasing RH
- Increasing CAPE
Summary

- Are cumulus updrafts **plumes** or **thermals**? Often a bit of both, encapsulated by the **thermal chain** structure (succession of rising thermals), which occurs over a wide range of conditions.

- Thermal chains occur in solutions to a simple equation set describing moist updrafts → intermediary regime in a continuum between the traditional thermal and plume/starting plume models.

- Evidence for **transition** from isolated thermal to thermal chain to starting plume structure with increasing $R^2/(zL)$, increasing environmental $RH$, and increasing $CAPE$. 
Summary, cntd.

- **Complicated entrainment behavior** → pulses of high fractional entrainment rate occur at the bottom of individual rising thermals in a chain. Perhaps this helps explain the challenge of obtaining simple entrainment rate scalings?

- **Locally enhanced entrainment** contributes significantly to overall *cloud dilution*, while local cloud regions can remain relatively *undilute*.

- Theory and LES indicate local enhancement in entrainment rate of factor ~2 consistent with difference of fractional entrainment between dry thermals and plumes from lab studies (Morton et al. 1956; Scorer 1957).
  - Probably not a coincidence, though mechanisms of entrainment appear to be quite different for *moist* convection than *dry*. 
Implications for modeling

• Is this work relevant to convection schemes?
  - I think so…?

• How can these ideas be implemented into schemes?
  - I have no idea…

• Because models in the “gray zone” under-resolve updrafts and hence over-predict $R^2/(ZL)$, we might expect them to produce *plume-like updrafts* contributing to *under-dilution*.

Varble et al. (2014)
Current and future work

- Role of environmental shear (Peters et al. 2019 JAS)
- Detrainment and downdrafts
- Observations of dynamics (Doppler or radar wind profiler)
- Large-domain, multi-cloud LES with “natural” convective initiation
- Detailed analysis (theory + LES) of interactions between thermal flow structure, ascent and volume growth rates, and entrainment/detrainment → *why is $R$ nearly constant for moist thermals but increases sharply for dry ones?*
- What controls $R$? (lots of relevant length scales…)
Thank you!

Questions?

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https://www.flickr.com/photos/nicholas_t/543334336
Conceptual model for “thermal chains”

Morrison et al. (2019), JAS (submitted)

Sustained ascent to LFC by buoyant pressure forcing induced by updraft’s B perturbation
LES of growing cumulus convection – sheared flow
- Two critical features for *moist* convection not accounted for by these conceptual models:
  - Increase of buoyancy from condensation and latent heating aloft
  - Decrease of buoyancy from entrainment and evaporation

- Modification to account for these challenges, e.g.,
  - separation of *dynamic* and smaller-scale *turbulent* entrainment
  - buoyancy sorting
“Commonly, these clouds were found to have life-spans of the order of one hour and a pulsating growth habit similar to that described in Scorcer and Ludlam’s (1953) bubble theory of convection. Each bubble or turret comprising the uppermost portion of the cloud was visible generally for 5 to 10 min: initially, as an active, hard appearing, ascending cloud mass; later, as a dissipating fibrous cloud mass whose place at the cloud summit was soon to be lost to a younger, active bubble.”
Schematic of the theoretical analytic model

Morrison (2017), JAS, Morrison et al. (2019), JAS (submitted)
Implied fractional entrainment rate $\varepsilon$

Profiles of analytic $\varepsilon$ for various $R$ and environmental $R_H$.

A complicated structure of $\varepsilon$, even in this simple analytic model!
Implied fractional entrainment rate $\varepsilon$

Profiles of analytic $\varepsilon$ for various $R$ and environmental $R_H$.

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Moser and Lasher-Trapp (2017), JAS
“Commonly, these clouds were found to have life-spans of the order of one hour and a pulsating growth habit similar to that described in Scorer and Ludlam’s (1953) bubble theory of convection. Each bubble or turret comprising the uppermost portion of the cloud was visible generally for 5 to 10 min: initially, as an active, hard appearing, ascending cloud mass; later, as a dissipating fibrous cloud mass whose place at the cloud summit was soon to be lost to a younger, active bubble.”
Initial bubble = 1000 m

Initial bubble = 1500 m
Tracer profiles

- b. $R = 600$, RH = 42.5%
- c. $R = 800$, RH = 42.5%
- e. $R = 1200$, RH = 42.5%
- f. $R = 1400$, RH = 42.5%
- h. $R = 1800$, RH = 42.5%
- i. $R = 2000$, RH = 42.5%

*Fig. 21. As in Fig. 18, but for the runs with an initial bubble radius of 2000 m.*
Peters et al. (2019), JAS (accepted)
In weakly sheared environments the *dynamic* perturbation pressure is fairly symmetric between the top and bottom of updrafts.

However, the *buoyant* perturbation pressure is *not* (it’s closer to being anti-symmetric)!

**Idealized 3D simulations using CM1 with an unsheared environment**

*Morrison (2016b) JAS, similar to Markowski and Richardson (2010)*
This implies that we can estimate $w_{\text{max}}$ including perturbation pressure effects by integrating the $w$ momentum equation using $p \sim p_B$! Actually works well for sheared flow too…

Thus, we can integrate $\frac{Dw}{Dt} \approx -\rho^{-1} \frac{\partial p_B}{\partial z} + B \equiv B_{\text{eff}}$ to estimate $w_{\text{max}}$.

Peters (2016) JAS

Idealized squall line simulation, 0-1.5 km $\Delta u$ of $\sim 19$ m/s.

Even in a unsheared environments, $p_D$ is essential to explain where $w_{\text{max}}$ occurs relative to the buoyancy profile.
A simple analytic model for $p_B$...

1) Derive a theoretical scaling of $p_B$ and $w$ based on approximate analytic solutions to the governing momentum and continuity equations assuming Boussinesq flow:

- 2D slab and axisymmetric cylindrical coordinates are used to compare 2D versus “3D” updrafts

- Buoyancy distributions are calculated from real and idealized soundings with various $R$ → *entrainment is not explicitly included*

2) Analytic solutions are compared to direct numerical solutions of the Poisson $p_B$ equation and (steady state) vertical velocity at the updraft center for the same buoyancy distributions.

*Morrison (2016a,b), JAS*
### 3D

\[
w_{max} = \sqrt{\frac{2CAPE}{1 + \frac{2\alpha^2R^2}{H^2}}} \\
\alpha \equiv \frac{\bar{w}}{w_0} = \frac{1}{w_0} \int_0^{2\pi} \int_0^R \frac{w}{\pi R^2} r dr d\theta
\]

For \( R \to 0 \):

\[w_{max} \to \sqrt{2CAPE}\]

For \( R \to \infty \):

\[w_{max} \to 0\]

Hydrostatic regime \((\alpha R/H)^2 \gg 1\):

\[w_{max} \approx \frac{H}{\alpha R} \sqrt{2CAPE}\]

### 2D

\[
w_{max} = \sqrt{\frac{2CAPE}{1 + \frac{8\alpha^2R^2}{H^2}}} \\
\alpha \equiv \frac{\bar{w}}{w_0} = \frac{1}{w_0} \int_0^R \frac{w}{R} dr
\]

For \( R \to 0 \):

\[w_{max} \to \sqrt{2CAPE}\]

For \( R \to \infty \):

\[w_{max} \to 0\]

Hydrostatic regime \((\alpha R/H)^2 \gg 1\):

\[w_{max} \approx \frac{H}{2\alpha R} \sqrt{2CAPE}\]
Direct numerical solution
(similar to Parker 2010)

\[ \nabla^2 p_B = \frac{\partial (B)}{\partial z} \]

W-K idealized sounding
(Weisman and Klemp 1982)

Horizontal buoyancy distribution specified as cosine function from updraft center to edge.

Integrate rising parcel from bottom to top using the calculated \( p_B \) field to obtain “numerical” \( w \) at the updraft center (\( r = 0 \)).
Comparison of $w_{\text{max}}$

Results shown for 6 different vertical buoyancy distributions based on various soundings.
Can this help explain the factor of 2 over-prediction of $w_{max}$ from $\sqrt{2CAPE}$?

Analytic expression for $w$ at height $z_m$

$\sqrt{2CAPE}$ (at different heights)

Effects of mixing/dilution at small $R$

Effects of perturbation pressure at large $R$
A rule of thumb is about a factor of 2 overestimation of $w_{\text{max}}$ from neglecting perturbation pressure effects and entrainment/mixing...

$\text{CAPE} \sim 4200 \text{ J kg}^{-1} \rightarrow w_{\text{max}} \sim 130 \text{ m s}^{-1}$
Hydrostatic regime \((\alpha R/H)^2 >> 1:\)

3D

\[
p = \frac{0}{H^2} \frac{2}{H^2} w_{LNB} = 2 \frac{0}{H^2} \left(1 + 2 \frac{2}{H^2} \right)^1 \text{CAPE}
\]

For \(R \to 0:\)

\[
p \approx p_h = 0 \text{CAPE}
\]

2D

\[
p = 4 \frac{0}{H^2} \frac{2}{H^2} w_{LNB} = 8 \frac{0}{H^2} \left(1 + 8 \frac{2}{H^2} \right)^1 \text{CAPE}
\]

For \(R \to 0:\)

\[
p \approx p_h = 0 \text{CAPE}
\]
Convection initiated using warm bubbles with different radii. Simulations run using the CM1 model (Bryan and Fritsch 2002). Results are calculated from 400-1080 sec.

Comparison with fully dynamical “updraft” simulations:

<table>
<thead>
<tr>
<th>Initial warm-bubble radius (km)</th>
<th>$R$ (km)</th>
<th>$H$ (km)</th>
<th>CAPE (J kg$^{-1}$)</th>
<th>$\Delta p$ (hPa)</th>
<th>Max $w$ (m s$^{-1}$)</th>
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</tbody>
</table>

Results are calculated from 400-1080 sec.
Buoyancy and $w$ at the center of the first thermal

Analytic

Numerical
The *plume* and *thermal* models originated from applying dimensional analysis to idealized flows*:

- **plumes** $\rightarrow$ buoyant jet in which buoyancy is supplied from a steady point source
- **thermal** $\rightarrow$ discrete rising buoyant bubble generated from a pulse source of buoyancy

*See Emanuel (1994)*
The *plume* and *thermal* models originated from applying dimensional analysis to idealized flows*:

- **plumes** \(\rightarrow\) buoyant jet in which buoyancy is supplied from a steady point source
- **thermal** \(\rightarrow\) discrete rising buoyant bubble generated from a pulse source of buoyancy

• Theoretical scalings are well-supported by lab studies, e.g. entrainment \(\sim 0.2/R\) in steady plumes.

• Large differences in flow characteristics between plumes and thermals (e.g. entrainment, vertical velocity structure)

*See Emanuel (1994)*
• Two critical features for moist convection not accounted for by these models:

  - Increase of buoyancy from condensation and latent heating aloft
  - Decrease of buoyancy from entrainment and evaporation

• Modification of traditional models to account for these challenges, e.g.,

  - separation of dynamic and smaller-scale turbulent entrainment
  - buoyancy sorting
• The earliest convection parameterizations and most current ones are based on the *steady state plume* framework

- **straightforward** to implement

- unlike the simple $R^{-1}$ scaling of entrainment in the traditional plume model, current schemes use a wide variety of methods to formulate entrainment (and detrainment) rates (functions of $w$, $z$, $RH$, etc.)
The earliest convection parameterizations and most current ones are based on the *steady state plume* framework:

- **straightforward** to implement

- unlike the simple $R^{-1}$ scaling of entrainment in the traditional plume model, current schemes use a wide variety of methods to formulate entrainment (and detrainment) rates (functions of $w$, $z$, $RH$, etc.)

→ no consistent scaling relationship has been found for entrainment rate in moist convection!
Observations of moist updrafts comprised of “bubbles” (thermal-like structures) go back at least to Scorer and Ludlam in the 1950’s.

More recent examples: Blyth et al. (1988), Damiani et al. (2006)

Recent LES ($\Delta x \sim 100$ m) support a thermal-like view, even for deep convection…

Varble et al. 2014
• The flow within individual thermals in high-resolution LES resembles Hill’s analytic spherical vortex (Sherwood et al. 2013; Romps and Charn 2015):

CM1 simulations ($\Delta x = 100$ m) with no environmental shear

![Streamlines; Hill’s vortex](image)

• Hill’s vortex is non-buoyant, extensions including the effects of buoyancy were derived by Morrison and Peters (2018), JAS.
• The **thermal** structure of updrafts challenges the steady-state **plume** framework assumed by most parameterizations
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A combination of the two: the *starting plume*...

Turner 1962, *JFM*
However... the starting plume model is not very satisfying because the wake behind the thermal head often itself consists of thermal-like structures. On the other hand, the rising thermal model is also unsatisfactory because the thermals that comprise a convective cloud are typically well organized.
Play timelapse movie
Vertical velocity retrievals from radar 1290 mHz profiler, Manaus, Brazil, Nov. 22 2014

see Giangrande et al. (2016) JGR for retrieval/deployment details
The succession of rising thermals is potentially a key aspect:

- Implications for entrainment/detrainment
- Implications for perturbation pressure structure
- Microphysics-dynamics interactions (Moser and Lasher-Trapp 2017)
To briefly summarize…
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Observations and LES suggest moist convection often occurs as a succession of rising thermals → “thermal chain”

Neither *thermal, plume, nor starting plume* models adequately describe thermal chains…
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Neither *thermal, plume, nor starting plume* models adequately describe thermal chains…

1. *Why* does this structure occur, and *what* are the driving mechanisms?

2. Can the behavior be described by simple scalings?

3. Are there linkages to the traditional plume or thermal models?
A theory for thermal chains*

*Morrison (2017), JAS
Morrison, Peters, Varble, Giangrande, Hannah, in prep.
Governing equations in axisymmetric coordinates \((r, z)\)*

\[
\frac{\partial \vec{u}}{\partial t} = -\vec{u} \frac{\partial u}{\partial r} - \vec{w} \frac{\partial u}{\partial z} - \rho_0^{-1} \left( \frac{\partial p}{\partial r} \hat{i} + \frac{\partial p}{\partial z} \hat{k} \right) + B \hat{k} \quad (1)
\]

\[
\frac{1}{r} \frac{\partial (ru)}{\partial r} + \frac{\partial \vec{w}}{\partial z} = 0 \quad (2)
\]

\[
\frac{\partial \theta}{\partial t} = -\vec{u} \frac{\partial \theta}{\partial r} - \vec{w} \frac{\partial \theta}{\partial z} + \frac{L_v}{c_p} CE \quad (3)
\]

\[
\frac{\partial q_v}{\partial t} = -\vec{u} \frac{\partial q_v}{\partial r} - \vec{w} \frac{\partial q_v}{\partial z} - CE \quad (4)
\]

\[
CE = \frac{\partial q_v}{\partial t} - \frac{\partial \theta}{\partial t} \frac{dq_s}{d\theta} \quad \text{if} \ q_v = q_s; \ CE = 0 \text{ otherwise} \quad (5)
\]

*Boussinesq, inviscid*
Analytic approximation

• Reynolds averaging applied, lateral turbulent fluxes represented by first-order Smagorinsky-type approach.

• Vertical turbulent fluxes are neglected.

• Condensate loading neglected in buoyancy, and within cloudy updrafts it is assumed that $q_v = q_s$ and sufficient cloud water is always available to retain saturated conditions when evaporation occurs.

• Solutions are first obtained for a scalar $C$, $B$, and $w$ at $r = 0$ and height of maximum $w$, $z_m$, for the primary ascending thermal, next at the thermal bottom, $z_{bot}$, then for additional thermals below in the chain.
Schematic of the analytic thermal chain model

\[ \Delta r = \frac{R}{2} \]

\[ \left( \frac{\partial C}{\partial r} \right)_U \approx \left( C_B \quad C_U \right) / r \]

Enhanced dynamic entrainment
Idealized WRF simulations (NOTE: not LES)

- 100 m horizontal grid spacing, enhanced Smagorinsky-type sub-grid scale mixing (mixing length $L = 500$ m)

- No microphysics except cloud condensation/evaporation, no condensate loading for simplicity

- Weisman-Klemp sounding, modified to have constant $RH$ above the level of free convection, unsheared environment

- Passive tracer added (held fixed at 1 below the LFC, initial values are zero above LFC)

Initial bubble width: 0.5, 1, 2, 4 km

Initial environmental $R_H$: 42.5 and 85%
$R_H = 85\%$

$t = 6\text{ min}$

$R_H = 42.5\%$

$t = 7\text{ min}$

$t = 9\text{ min}$

$t = 10\text{ min}$

$t = 12\text{ min}$

$t = 13\text{ min}$
Evolution of a passive scalar at $z_m$

The analytic model:

- Organized horizontal advection is zero at updraft center by symmetry ($u = 0$ at updraft center $r = 0$)

- Horizontal convergence across updraft is 0 at height of maximum $w$ ($z = z_m$)

- Assume $C = 0$ in the environment

- $R$ is calculated directly from the simulations
Comparison of analytic and numerical solutions

\[ C_z = C_0 e^{\frac{2k^2Lz}{P_rR^2}} \frac{\left(1 + \frac{k^2Lz}{(P_rR^2)}\right)}{\left(1 + k^2Lz/(P_rR^2)\right)} C_0 \]  

Analytic solution

\[ R_H = 42.5\% \]
Buoyancy at $z_m$

Similar procedure as scalar mixing, but accounts for $\frac{\partial q_v}{\partial t}$ and $\frac{\partial \theta}{\partial t}$:

1) mixing of buoyancy itself (assuming $B = 0$ in the environment)

2) mixing of dry air from the environment

\[
B = \frac{B_{AD}}{1 + \frac{k^2 L z}{P_r R^2 c_p \left( \frac{P_r R^2}{z} + k^2 L \right)}} \frac{2 L_v g k^2 L}{c_p} \left( \frac{P_r R^2}{z} + k^2 L \right) = \frac{1}{z} \int_{z_{LFC}}^{z} q_{SE} \left( \frac{R_H}{T_E} \right) dz
\]
Vertical velocity at $z_m$

- Similar procedure compared to scalar mixing, but accounts for:
  1) dilution of buoyancy
  2) dilution of momentum
  3) buoyant perturbation pressure effects

\[
\begin{align*}
\frac{\text{\textit{\texttimes k^2L}}}{{R^2\text{CAPE}}}\left(\frac{P_rR^2z}{k^2L} + \frac{P_r^2R^4}{k^4L^2}\ln\left(\frac{k^2Lz}{P_rR^2} + 1\right)\right) \\
\left(1 + \frac{2R^2}{H^2} + \frac{4k^2Lz}{3R^2}\right)
\end{align*}
\]
Now for $z_{bot}$...

$\Delta r = R/2$

$\frac{\partial C}{\partial r}_U \approx \left( C_B \ C_U \right) / r$

Enhanced dynamic entrainment
Assuming a linear horizontal profile of $u$ such that $u_U \sim u_B/2$ (since $u = 0$ at $r = 0$), $C_B = 0$, combining with the horizontally-averaged mass continuity equation, approximating vertical derivatives as $\frac{\partial \varphi}{\partial z} \approx \frac{\varphi_{z} - \varphi_{LFC}}{z - z_{LFC}}$, and with boundary conditions $w_U = 0$ and $C_U = C_0$ at the LFC gives

$$C_U \frac{2}{3} C_0$$

All else equal, this suggests an increase in horizontal gradients between $r = 0$ and $r = R/2$ at the thermal bottom by a factor of 3/2 that is self-similar, i.e. does not depend on $R$, $w$, $z$ – this increases lateral mixing of environmental and updraft air at $r = 0$ and $z_{bot}$
Assuming a similar scaling applies to $w$ (though not a passive scalar) gives an increase in the lateral turbulent mixing of $9/4$ at $r = 0$, since mixing is proportional to $\frac{\partial C}{\partial r} \left| \frac{\partial w}{\partial r} \right|$. Similar approach is applied to give $B_{bot}$ and $w_{bot}$...

$$C = \frac{1 - 9k^2Lz/(4PrR^2)}{1 + 9k^2Lz/(4PrR^2)} C_0$$

$$B = \frac{B_{AD}}{1 + \frac{9k^2Lz}{4PrR^2_{HMB}}} \frac{9L_v g k^2 L}{2c_p \left( \frac{Pr^2_{HMB}}{z} + \frac{9k^2L}{4} \right)}$$

$$w = \sqrt{2\text{CAPE} \left[ \left( \frac{1}{z^2} \frac{9L_v g k^2 L}{4c_p Pr^2_{HMB} \text{CAPE}} \right) \left( \frac{4Pr^2_{HMB}z}{9k^2L} \frac{16Pr^2_{HMB}}{81k^4L^2} \ln \left( \frac{9k^2Lz}{4Pr^2_{HMB}} + 1 \right) \right) \right]}$$
Enhanced mixing and reduced $B$ and $w$ at the thermal bottom in turn leads to reduced (or even negative) $\partial w/\partial z$ below the thermal. With approximation, this is captured with a similar finite differencing to give $B$ and $w$ at $z_{bel}$:

$$C = \frac{1 - \xi k^2 L z / (P_r R^2)}{1 + \xi k^2 L z / (P_r R^2)} C_0$$

at the

$$\sim \left[ \left( \frac{w_L}{z_{bel}} \right)^{-1} \left( \frac{w_L}{z_{bel}} + \frac{w_{bot}}{z_{bot}} \right) \right]^2$$

$$B_{bel} = \frac{B_{AD}}{1 + \xi k^2 L z / (P_r R^2)} - \frac{2 \xi L v g k^2 L \Phi}{c_p (P_r R^2 / z + \xi k^2 L)}$$

$$B_{bel} = \frac{B_{AD}}{1 + \xi k^2 L z / (P_r R^2)} - \frac{2 \xi L v g k}{c_p (P_r R^2 / z)}.$$

Method is repeated for multiple thermals…
Comparison with numerical modeling (axisymmetric CM1)

- 100 m grid spacing (vertical and horizontal), enhanced Smagorinsky-type sub-grid scale mixing (horizontal mixing length $L = 500$ m)

- Convection initiated with applying warm bubbles of varying radius (600 to 2000 m)

- Environmental (above level of free convection) RH of 42.5% or 85%

- Weisman-Klemp initial sounding (except RH changes), no environmental shear

- Passive tracer set to 1 in the lowest 1.5 km
Comparison of numerical and analytic solutions

Analytic
Numerical

Passive Tracer

Results for each panel are when $z_m \sim 6$ km.

RH=0.85
(but similar results for RH=0.425)
Comparison of numerical and analytic solutions

**Buoyancy**

| RH = 0.425 |
| R = 0.5 km |
| Height (m) | RH = 0.425 |
| B (m s⁻¹) |
| R = 1.0 km |
| Height (m) | RH = 0.425 |
| B (m s⁻¹) |
| R = 1.4 km |
| Height (m) | RH = 0.425 |
| B (m s⁻¹) |
| R = 1.8 km |
| Height (m) | RH = 0.425 |
| B (m s⁻¹) |
| RH = 0.85 |
| R = 0.6 km |
| Height (m) | RH = 0.85 |
| B (m s⁻¹) |
| R = 1.0 km |
| Height (m) | RH = 0.85 |
| B (m s⁻¹) |
| R = 1.4 km |
| Height (m) | RH = 0.85 |
| B (m s⁻¹) |
| RH = 0.85 |
| R = 1.8 km |
| Height (m) | RH = 0.85 |
| B (m s⁻¹) |
| R = 2.0 km |
| Height (m) | RH = 0.85 |
| B (m s⁻¹) |

**Vertical Velocity**

| RH = 0.425 |
| R = 0.5 km |
| w (m s⁻¹) |
| Height (m) |
| R = 1.0 km |
| Height (m) |
| R = 1.4 km |
| Height (m) |
| R = 1.8 km |
| Height (m) |
| RH = 0.85 |
| R = 0.6 km |
| Height (m) |
| R = 1.0 km |
| Height (m) |
| R = 1.4 km |
| Height (m) |
| R = 1.8 km |
| Height (m) |

**Analytic vs. Numerical**

RH = 0.425
RH = 0.85
Main findings:

1. Analytic and numerical models show a continuum behavior between thermal and plume structures, determined by updraft $zL/R^2$, environmental $RH$, and CAPE.

2. The thermal chain is a transitional regime between plumes and thermals, has features of both but also unique characteristics. Occurs over a wide range of conditions.

3. The thermal chain structure arises directly from solutions of the governing equations for moist convection. Results suggest interactions between entrainment, evaporation, buoyancy, and flow structure are critical for thermal chains.
EXTRA
• Increasing **CAPE** $\rightarrow$ transition from **isolated thermal** to **thermal chain** to **plume** occurs at smaller $R$.

• Interactions between entrainment, evaporation, buoyancy, and flow are **critical** for thermal chain structure.
By extending Hill’s vortex solution to include buoyancy, we derive an analytic expression for the ratio of thermal ascent rate and maximum vertical velocity, \( \lambda \), that is a quadratic function of two non-dimensional buoyancy parameters (Morrison and Peters 2018, JAS).

**Numerically simulated vs. theoretical ratio of thermal ascent rate and maximum vertical velocity within the thermal**

Hill’s vortex \( \lambda = 0.4 \)

**Numerically simulated vs. theoretical thermal ascent rate**

![Graph showing numerical simulation vs. theoretical ratio and maximum vertical velocity](image)
Analytic vertical velocity as a function of $R$

($R_H = 50\%, L \sim R$)
Hill’s vortex-like flow in the upper turret with maximum $w$ in the vortex center at height $z_m$.

$u \sim 0$ near $z_m \rightarrow$ limited impact of dynamic entrainment from organized convective-scale flow at this height.

Below $z_m$ dynamic entrainment is important $\rightarrow$ inflow of environmental air leads to engulfment and mixing, decreasing $R$ and sharpening horizontal gradients – this in turn increases lateral turbulent mixing at the updraft center.
$R_H = 85\%$

$t = 7\; \text{min}$

$R_H = 42.5\%$

$t = 8\; \text{min}$

$t = 10\; \text{min}$

$t = 11\; \text{min}$

$t = 13\; \text{min}$

$t = 14\; \text{min}$