An introduction to numerical continuation with AUTO

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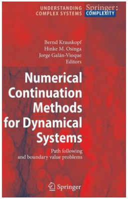
AUTO

Software for continuation and bifurcation problems in ordinary differential equations.

- Originally developed by Eusebius Doedel, with contributions from many others.
- For more information and to download AUTO visit http://cmvl.cs.concordia.ca/auto [lecture notes and manual]
- For more info on the theory behind continuation [Krauskopf, Osinga & Galán-Vioque (Eds.) Springer, 2007]



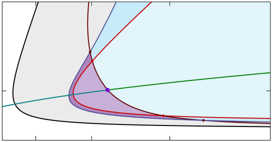
Related software includes:
DSTool, PyDSTool, XPPAUT, Content, MatCont, and DDE-BifTool.

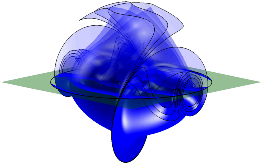
ODE overview

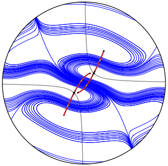
$$u'(t) = f(u(t), \lambda), \quad f, u \in \mathbb{R}^n,\ \lambda \in \mathbb{R}$$

- Compute families of stable and unstable periodic solutions
- Locate and continue folds, branch points, period doubling bifurcations, and bifurcations to tori.
- Do each of the above for rotations
- Follow homoclinic orbits and detect and continue codim-2 bifurcations, using HomCont.
- Compute curves of solutions on [0,1], subject to nonlinear boundary and integral conditions.
- Determine folds and branch points along solution families to the above BVP.

and more...







[Creaser et al. 2017]

Predator prey model

Consider a predator-prey model with fishing

$$\begin{cases} u_1' = 3u_1(1-u_1) - 2u_1u_2 - \lambda(1-e^{-5u_1}), \\ u_2' = -u_2 + 3u_1u_2. \end{cases}$$

We can think of u_1 as 'fish' and u_2 as 'sharks', while the term

$$\lambda(1-e^{-5u_1}),$$

represents 'fishing' with a 'quota' λ.

For $\lambda = 0$ there are three equilibria

$$(u_1, u_2) = (0,0),\ (1,0),\ \left(\frac{1}{3}, 1\right).$$

Implicit Function Theorem

Consider $\mathbf{f} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ with $\mathbf{f}(\mathbf{u}_0, \lambda_0) = \mathbf{0}$ for $\mathbf{u}_0 \in \mathbb{R}^n$ and $\lambda_0 \in \mathbb{R}$. Suppose the following holds:

- The Jacobian matrix $\mathbf{f}_\mathbf{u}(\mathbf{u}_0, \lambda_0)$ is nonsingular;
- $\mathbf{f}$ and $\mathbf{f}_\mathbf{u}$ are Lipschitz continuous (in both $\mathbf{u}$ and λ)

Then there exists $\delta > 0$ and interval $\Lambda_\delta = (\lambda_0 - \delta, \lambda_0 + \delta)$, with a unique function $\mathbf{u}(\lambda)$ continuous on Λ_δ, such that

$$\mathbf{u}(\lambda_0) = \mathbf{u}_0$$
$$\mathbf{f}(\mathbf{u}(\lambda), \lambda) = \mathbf{0}, \text{ for all } ||\lambda - \lambda_0|| < \delta.$$

We call $\mathbf{u}_0$ *an* isolated *solution of* $\mathbf{f}(\mathbf{u}, \lambda_0) = 0$.

Jacobians

The Jacobian matrix for our example is

$$J(u_1, u_2; \lambda) = \begin{pmatrix} 3 - 6u_1 - 2u_2 - 5\lambda e^{-5u_1} & -2u_1 \\ u_2 & -1 + 3u_1 \end{pmatrix}.$$

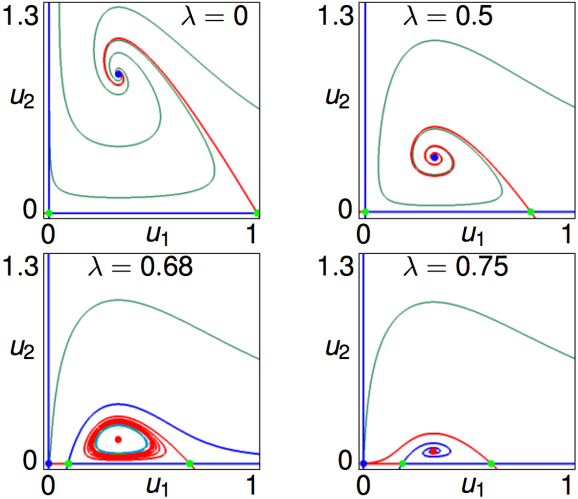
Evaluating at the equilibria gives

$$J(0,0;0) = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \; J(1,0;0) = \begin{pmatrix} 3 & -2 \\ 0 & 2 \end{pmatrix},$$

$$J(\frac{1}{3}, 1; 0) = \begin{pmatrix} -1 & -\frac{2}{3} \\ 6 & 0 \end{pmatrix}$$

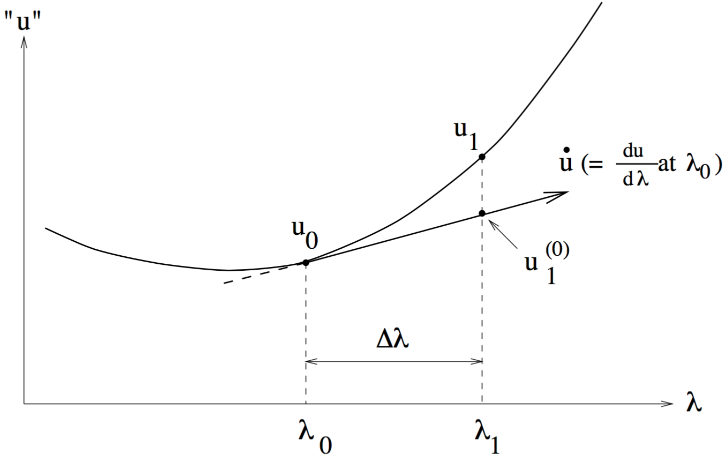
all three are non-singular for $\lambda = 0$. Therefore, by the IFT all three equilibria persist for (small) $\lambda \neq 0$

PHASE PORTRAITS



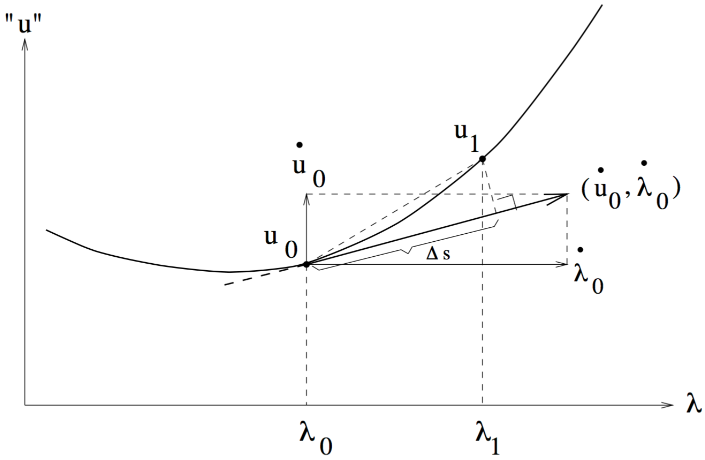
Parameter continuation

- Set $\lambda_1 = \lambda_0 + \Delta\lambda$
- Use Newton's method with initial guess $\mathbf{u}_1^{(0)} = \mathbf{u}(\lambda) + \Delta\lambda\dot{\mathbf{u}}$



Pseudo-arclength continuation

- $\mathbf{f}(\mathbf{u}_1, \lambda_1) = 0$
- $(\mathbf{u}_1 - \mathbf{u}_0)\dot{\mathbf{u}}_0 + (\lambda_1 - \lambda_0)\dot{\lambda}_0 - \Delta s = 0$



User supplied files

To run AUTO you will need:

- A source file `xxx.{f,f90,c}` containing the Fortran routines FUNC, STPNT, BCND, ICND, FOPT, and PVLS.
- A constants-file c.xxx.

See `auto/07p/demos` for lots of examples.

AUTO is run via the AUTO command line user interface (CLUI) - based on Python

AUTO Errors

Error MX

- Something in c. file? Problem with parameters, not initialised properly (not good maths!) or doesn't quite match BCs, check PVLS and BCs.

Error NaN

- Something in .f90 file?

Error ValueError: invalid literal for int() with base 10: 'x-'

- Missed a comma in the c.file

Error IndexError: string index out of range

- Some typo in your c. file - go back to an older one and try again.

Error write() argument must be str, not bytes

- in c. file if IRS greater than 0 then you get this error if using python3