Concentration inequalities in non-stationary chaotic deterministic dynamical systems

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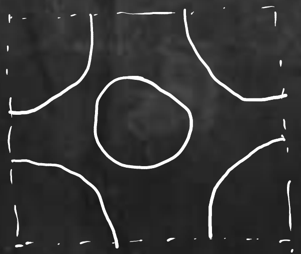
Intermittent maps (Pomeau-Manneville, Liverani-Saussol-Vaienti)

$f: [0,1] \to [0,1]$

$$x \mapsto \begin{cases} x(1+2^\gamma x^\gamma), & x \le 1/2 \\ 2x-1, & x > \frac{1}{2} \end{cases}$$

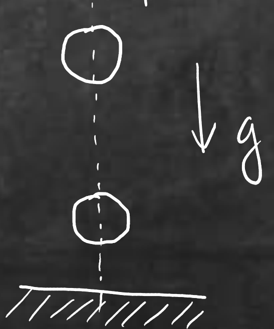
γ - parameter

Prototypical for more complex dynamical systems

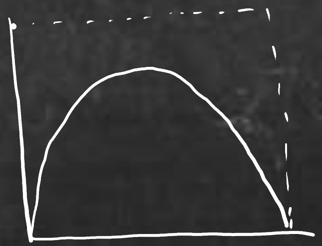


Sinai

Bunimovich



Wojtkowski



logistic

Hénon

Birkhoff sums and concentration inequalities

$v : [0,1] \to \mathbb{R}$ Lipschitz

$V_n = v + v\circ f + \dots + v \circ f^{n-1}$ — random variable / process on $([0,1], \mathbb{P})$, $\mathbb{P}$ = Lebesgue

$S_n = V - \mathbb{E} V_n$

How fast does S_n grow?

What is $\mathbb{P}(S_n \geq a_n)$ or $\mathbb{P}(\max_{k \leq n} S_k \geq a_n)$, if $a_n \to \infty$?

Example of concentration inequality: Hoeffding's inequality:

If $X_n \in [-\frac{1}{2}, \frac{1}{2}]$ i.i.d. with $\mathbb{E} X_n = 0$, then

$$\mathbb{P}\Big(\sum_{k \leq n} X_n \geq a\Big) \leq e^{-\frac{2a^2}{n}}.$$

too good for dynamical systems

Limit laws for intermittent maps (Gouëzel '04)

a) If $\gamma \in [0, \frac{1}{2})$, then $\frac{1}{\sqrt{n}} S_n \longrightarrow N(0, \sigma^2)$

Thus we know $\lim\limits_{n\to\infty} \mathbb{P}(S_n \geq a\sqrt{n})$

b) If $\gamma = \frac{1}{2}$, then $\frac{1}{\sqrt{n \log n}} S_n \longrightarrow N(0, \sigma^2)$

c) If $\gamma \in (\frac{1}{2}, 1)$, then $\frac{1}{n^\gamma} S_n \longrightarrow$ α-stable process, $\alpha = \frac{1}{\gamma}$

d) If $\gamma \geq 1$, then $\frac{1}{n} S_n \longrightarrow 0$

Simple concentration inequalities: moment bounds

If $\mathbb{E}|S_n|^p \leq M_n$, then $\mathbb{P}(|S_n| \geq a) \leq \frac{M_n}{a^p}$ — the larger p and smaller M_n, the better.

Thm (Gouëzel, Melbourne '14)

a) if $\gamma \in (0, \frac{1}{2})$ then $\|S_n\|_p \sim n^{\frac{1}{2}}$ for all $p < \frac{2}{\gamma} - 2$

b) if $\gamma = \frac{1}{2}$ then $\|S_n\|_p \sim n \log n$ for all $p < 2$

c) if $\gamma \in (\frac{1}{2}, 1)$ then $\|S_n\|_p \sim n\gamma$ for all $p < \frac{1}{\gamma}$

$\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$

Thm (K., Leppänen '20+) Same ↑ for non-stationary dynamics:

$\gamma_1, \gamma_2, \ldots \leq \gamma$ $\qquad V_n = v_0 + v_1 \circ F_1 + \cdots + v_{n-1} \circ F_{n-1}$

$F_n = f_{\gamma_n} \circ \cdots \circ f_{\gamma_1}$ $\qquad S_n = V_n - \mathbb{E} V_n$ $\qquad$ "changing environment"

Martingale approach

Def M_n is a martingale wrt filtration $\mathcal{F}_n$ if $M_n \in \mathcal{F}_n$ and

$$\mathbb{E}(M_{n+1} \mid \mathcal{F}_n) = M_n$$

$$[M]_n = \sum_{k \le n} |M_k - M_{k-1}|^2 \quad \text{— quadratic variation}$$

$$M_n^* = \max_{k \le n} |M_n| \quad \text{— record process}$$

Thm (Burkholder) For each $p \ge 1$ $\exists\, c_p, C_p > 0$ such that for all martingales M_n,

$$c_p \| [M]_n \|_{p/2}^{1/2} \le \| M_n^* \|_p \le C_p \| [M]_n \|_{p/2}^{1/2}$$

Similar result holds for Lorentz norms

$$\| \cdot \|_{p,\infty} \sim \left(\sup_{t>0} t^p\, \mathbb{P}(\cdot \ge t) \right)^{1/p}$$

$\underline{\text{Cor}}$ $\|M_n^*\|_p \le C_p \sum_{k\le n} \|M_k - M_{k-1}\|_p$

Where do we find a martingale?

Symbolic coding: let $x_n = F_n(x)$ be the trajectory and

$$a_n = \begin{cases} 0, & x_n < 1/2 \\ 1, & \text{else} \end{cases}$$

Filtration: $\mathcal{F}_n$ generated by $(a_0, \dots, a_{n-1})$

Doob martingale: $M_n = \mathbb{E}(S_N \mid \mathcal{F}_n)$ (N fixed)

How to bound $\|M_n - M_{n-1}\|_p$?

Look at $M_1 - M_0 = M_1 = \mathbb{E}(S_N \mid a_0)$

$$\mathbb{E}(S_N \mid a_0 = 1) - \mathbb{E}(S_N) = \int (v_0 + v_1 \circ F_1 + \cdots + v_{N-1} \circ F_{N-1})(d\mu - dm)$$

$$= \sum_{k<n} \int v_k \, d(F_k)_* (\mu - m)$$

where $m =$ Lebesgue$[0,1]$ and $\mu =$ Lebesgue$[\frac{1}{2}, 1]$ / normalization.

Memory loss

classic : $|(F_n)_*(\mu-\mu')| \lesssim n^{-(\frac{1}{\gamma}-1)}(\log n)^{\frac{1}{\gamma}}$

for "regular" measures μ, μ'

stationary: Young '99, Gouëzel '04

non-stationary: Aimino, Hu, Nicol, Török, Vaienti '15

unexpected : $|(F_n)_*(\mu-\mu')| \lesssim n^{-\frac{1}{\gamma}}$

for "special" measures μ, μ' (including those with Bounded Lipschitz densities)

stationary : Gouëzel '04 (Renewal theory, Wiener lemma)

Markov chains: Lindvall '79 (meeting time)

Lemma (K., Leppänen)

$$\|(F_n)_*(\mu-\mu')\| \lesssim n^{-1/\gamma}$$

in non-stationary case, for measures with Lipschitz densities (and more).

Proof $\exists\ \varkappa_n \geq 0$, $\sum_{n=1}^{\infty} \varkappa_n = 1$ and $\sum_{k=n}^{\infty} \varkappa_n \lesssim n^{-1/\gamma}$ such that every "good" measure μ has a decomposition

$\mu = \sum_{n=1}^{\infty} \varkappa_n \mu_n$, where $(F_n)_* \mu_n =$ Lebesgue.

$\blacksquare$

thank you