Rate-Induced Bifurcations: Can We Adapting to a Changing Environment?

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What might happen in a warmer world?

The ultimate objective is ... stabilisation of greenhouse gas concentrations in the atmosphere at a level that would prevent dangerous anthropogenic interference with the climate system...
... such a level should be achieved within a time-frame sufficient to allow ecosystems to adapt naturally to climate change, to ensure food production is not threatened, and to enable economic development to proceed in a sustainable manner.
The United Nations Framework Convention on Climate Change (UNFCCC)

dangerous levels and dangerous rates for the climate

Contents

Other real world systems

- The compost-bomb instability

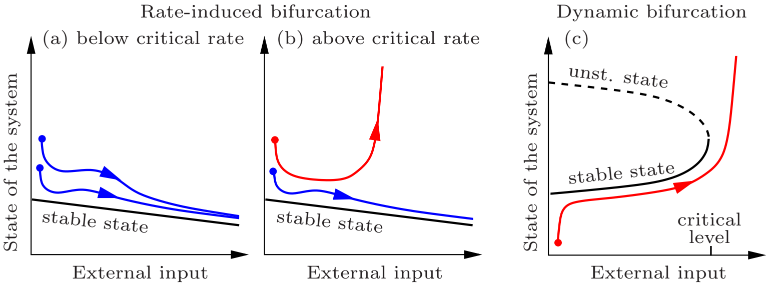
  C. Luke & P. Cox 2011 *Eur. J. Soil Sci.* **62**, 5-12.

- The thermo-haline circulation with freshwater forcing

  V. Lucarini, S. Calmanti & V. Artale 2005 *Climate Dynamics* **24**, 253-262.

- Type III neural excitability

  J. Mitry, M. McCarthy, N. Kopell & and M. Wechselberger 2013 *J. Math. Neuro.* **3**, 12.



A trajectory tracks the continuously changing stable state or destabilises.

The mathematical description

$$\frac{dx}{dt} = f(x, \lambda(\epsilon t))$$

x system variable;
λ external input;
ϵ rate of change.

For every fixed value of λ there is a stable state $\tilde{x}(\lambda)$.

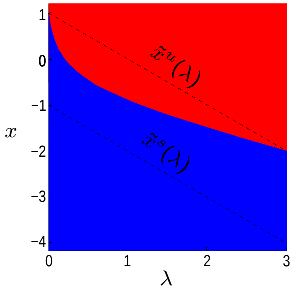
- A rate-induced bifurcation is a non-autonomous instability.
- ϵ_c associated with validity boundary of Fenichel's theorem.

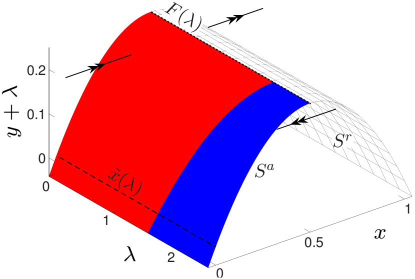
Questions

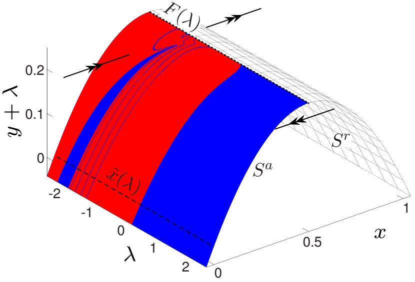
For different systems $f(x, \lambda(\epsilon t))$ and external inputs $\lambda(\epsilon t)$:

- Does the system have a critical rate ϵ_c?
- Can we compute ϵ_c?
- What is the threshold seperating initial conditions that track $\tilde{x}(\lambda(\epsilon t))$ from those that destabilise?

The obvious... and the non-obvious, both simple and complicated



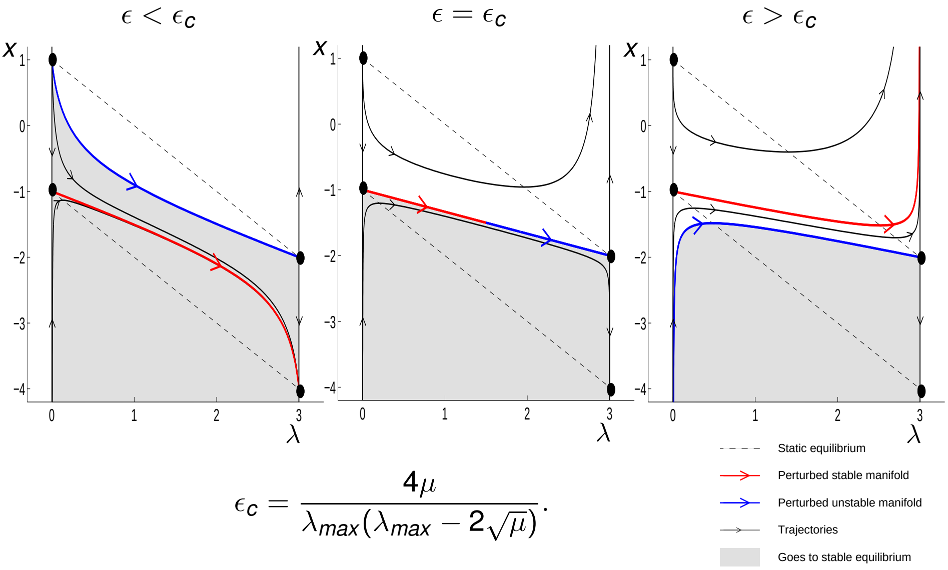




tracks or destabilises

A system with an obvious threshold

A system with an obvious threshold



$$\epsilon_c = \frac{4\mu}{\lambda_{max}(\lambda_{max} - 2\sqrt{\mu})}.$$

Thresholds in multiple timescale systems

$$0 < \delta \ll 1 \qquad \text{and} \qquad T = \delta t$$

$$\begin{aligned} \delta \frac{dx}{dt} &= y + \lambda + x(x-1) \\ \frac{dy}{dt} &= -x \end{aligned} \qquad\qquad \begin{aligned} \frac{dx}{dT} &= y + \lambda + x(x-1) \\ \frac{dy}{dT} &= -\delta x \end{aligned}$$

Thresholds in multiple timescale systems

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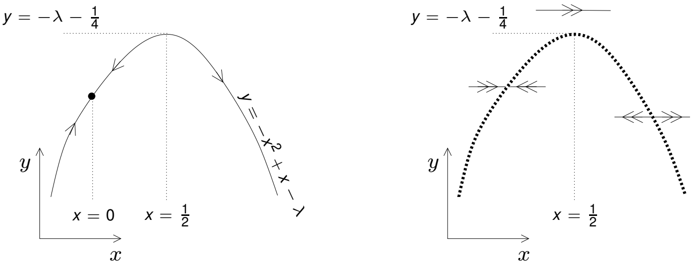

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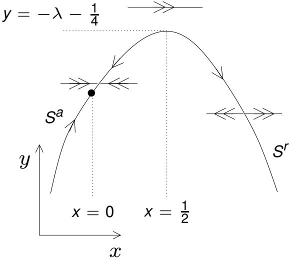
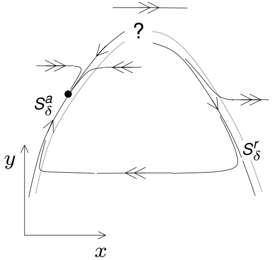
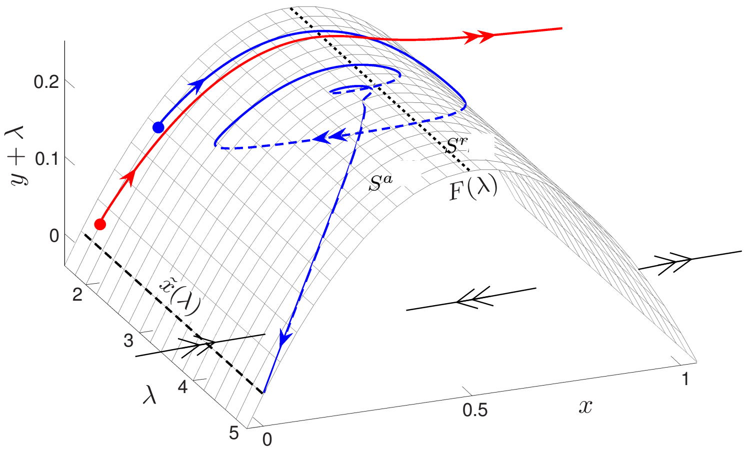
$\delta = 0$





Changing the external input

Thresholds in multiple timescale systems



Thresholds in multiple timescale systems

$$\begin{aligned}\delta\frac{dx}{dt} &= f(x,y,\lambda,\delta)\\ \frac{dy}{dt} &= g(x,y,\lambda,\delta)\\ \frac{d\lambda}{dt} &= \epsilon h(\lambda)\end{aligned}$$

Assumptions:

- There is a fold F tangent to the fast x direction: $\frac{\partial f}{\partial x} = 0$, $\frac{\partial^2 f}{\partial x^2} \neq 0$.
- For fixed λ there is unique single curve of stable states $\tilde{x}(\lambda)$ close to the fold.
- The critical manifold S can be expressed as a graph $y(x,\lambda)$.
- The system has 1 fast and 2 slow variables: $0 < \delta << \epsilon < 1$.

A closer look at the equations

$$\begin{aligned}
\delta\frac{dx}{dt} &= f(x,y,\lambda,\delta) \\
\frac{dy}{dt} &= g(x,y,\lambda,\delta) \\
\frac{d\lambda}{dt} &= \epsilon h(\lambda)
\end{aligned}$$

Project onto S:

$$\begin{aligned}
-\frac{\partial f}{\partial x}\bigg|_S \frac{dx}{dt} &= \left(g\frac{\partial f}{\partial y} + h\frac{\partial f}{\partial \lambda}\right)\bigg|_S \\
\frac{d\lambda}{dt} &= \epsilon h|_S
\end{aligned}$$

A closer look at the equations

$$\begin{aligned}
\delta \frac{dx}{dt} &= f(x, y, \lambda, \delta) \\
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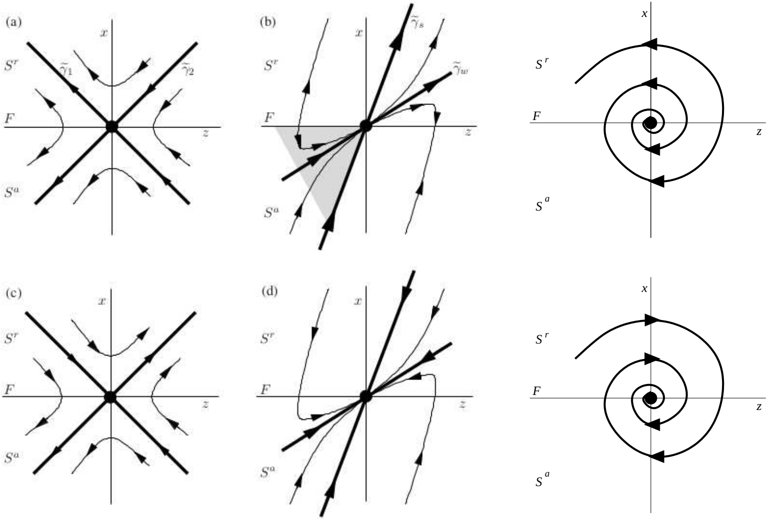
\frac{d\lambda}{dt} &= \epsilon h|_S
\end{aligned}$$

Desingularise by $dt = -ds(\partial f/\partial x)|_S$:

$$\begin{aligned}
\frac{dx}{ds} &= \left.\left(g\frac{\partial f}{\partial y} + h\frac{\partial f}{\partial \lambda}\right)\right|_S \\
\frac{d\lambda}{ds} &= -\epsilon \left.\left(h\frac{\partial f}{\partial x}\right)\right|_S
\end{aligned}$$

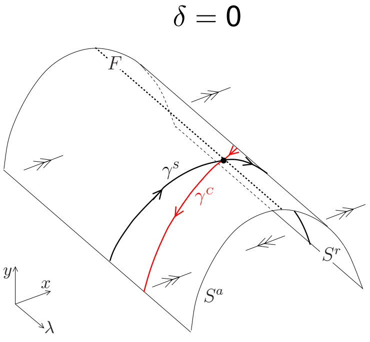
Classification of folded singlarities

In the desingularised system time is reverse on S^r.



Desroches et al, 2012

Along the fold F when $\delta \neq 0$



Existence results for multi-scale systems

Existence of critical rates:

There is a critical rate ϵ_c if for some point (x, y, λ) at time t both

$$\left(g\frac{\partial f}{\partial y} + h\frac{\partial f}{\partial \lambda} \right)\bigg|_F = 0$$

and

$$\frac{d}{d\epsilon} \left(g\frac{\partial f}{\partial y} + h\frac{\partial f}{\partial \lambda} \right)\bigg|_F \neq 0.$$

Existence of non-obvious thresholds:

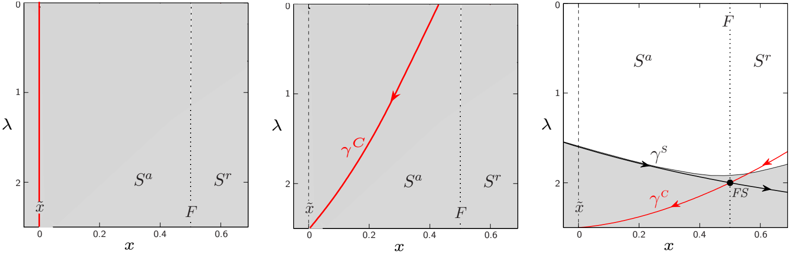
An instability threshold requires a folded saddle singularity. Moreover, the system is guaranteed to have an instability threshold, if a folded saddle is the only folded singularity.

Simple threshold with an isolated folded saddle

$$\delta \frac{dx}{dt} = y + \lambda + x(x-1); \qquad \frac{dy}{dt} = -x; \qquad \frac{d\lambda}{dt} = \epsilon(\lambda_{\max} - \lambda).$$

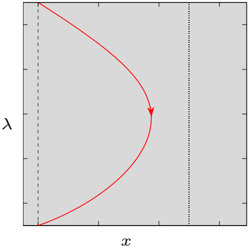
$$\epsilon_c = \frac{1}{\lambda_{\max}}.$$

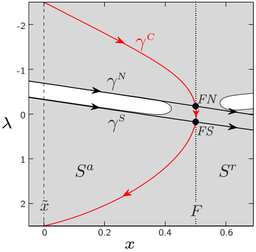
$\epsilon = 0$ $\quad$ $\epsilon < \epsilon_c$ $\quad$ $\epsilon > \epsilon_c$



Complicated threshold after a folded saddle-node bifurcation

$$\delta\frac{dx}{dt} = y + \lambda + x(x-1); \qquad \frac{dy}{dt} = -x; \qquad \frac{d\lambda}{dt} = \epsilon(\lambda_{\max}^2 - \lambda^2)/\lambda_{\max}.$$

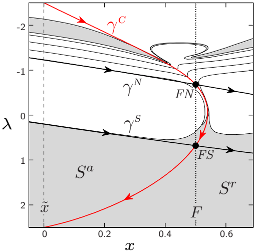


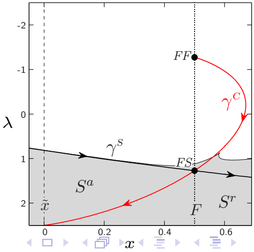


$$\epsilon_c = \frac{1}{\lambda_{\max}}.$$

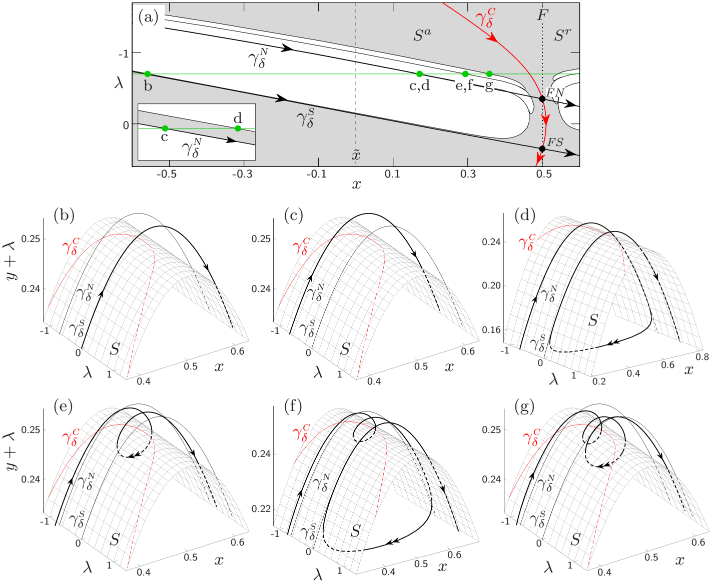
Panel 1: $\epsilon < \epsilon_c$

Panels 2-4: $\epsilon > \epsilon_c$





Composite canards form the boundary



Conclusions

Rate-induced bifurcations are of key importance in many real-world system. We seek to identify critical rates and instability thresholds in a range of systems.

Obvious rate-induced bifurcations: find a hetroclinic connection.

Non-obvious rate-induced bifurcations: use geometric singular perturbation theory, find folded singularities and canards:

- Existence results for critical rates and instability thresholds.
- Find a new complicated banded threshold.
- Identify the structure with new composite canards.

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