Rate-Induced Bifurcations: Can We Adapting to a Changing Environment?

Part II

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Real world systems

- The compost-bomb instability

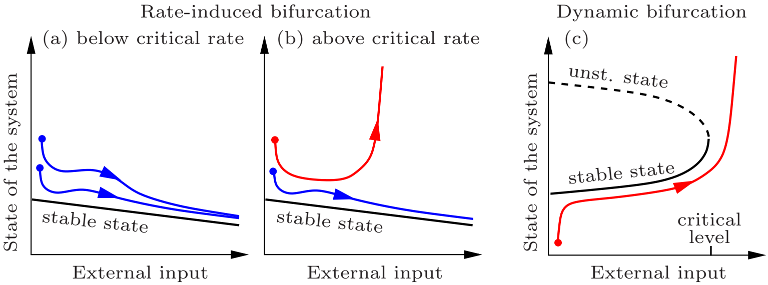
  C. Luke & P. Cox 2011 *Eur. J. Soil Sci.* **62**, 5-12.

- The thermo-haline circulation with freshwater forcing

  V. Lucarini, S. Calmanti & V. Artale 2005 *Climate Dynamics* **24**, 253-262.

- Type III neural excitability

  J. Mitry, M. McCarthy, N. Kopell & and M. Wechselberger 2013 *J. Math. Neuro.* **3**, 12.



A trajectory tracks the continuously changing stable state or destabilises.

The mathematical description

$$\frac{dx}{dt} = f(x, \lambda(\epsilon t))$$

x system variable;
λ external input;
ϵ rate of change.

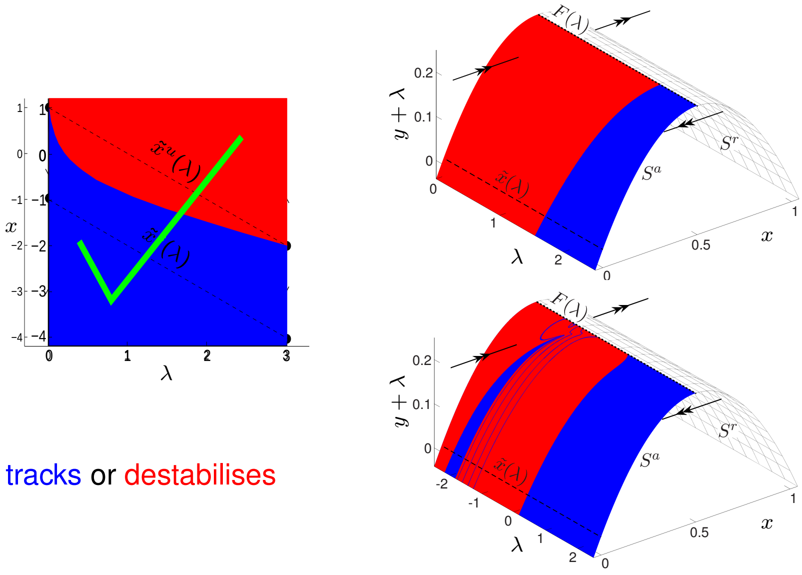
For every fixed value of λ there is a stable state $\tilde{x}(\lambda)$.

Questions

For different systems $f(x, \lambda(\epsilon t))$ and external inputs $\lambda(\epsilon t)$:

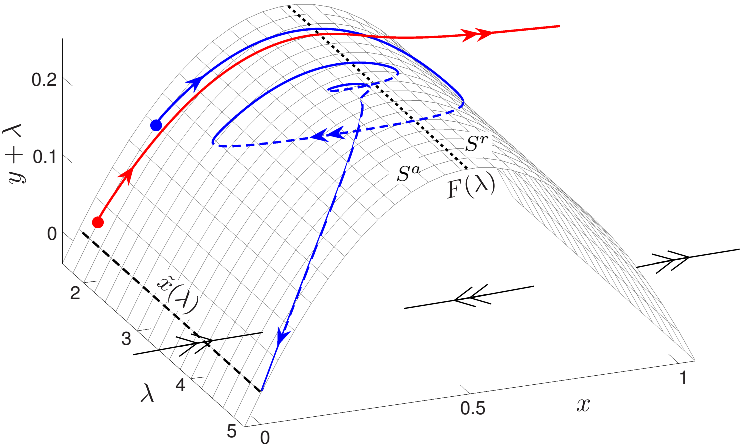
- Does the system have a critical rate ϵ_c?
- Can we compute ϵ_c?
- What is the threshold seperating initial conditions that track $\tilde{x}(\lambda(\epsilon t))$ from those that destabilise?

The obvious... and the non-obvious, both simple and complicated



tracks or destabilises

Thresholds in multiple timescale systems



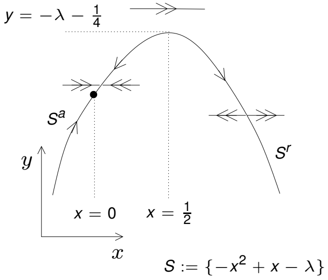
A trajectory tracks $\tilde{x}(\lambda)$
or destabilises and moves away in the fast x-direction.

Analysing multiple timescale systems

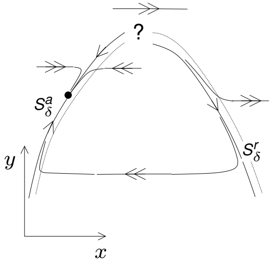
Fix λ, $0 < \delta \ll 1$ and $t = \delta T$.

$$\delta \frac{dx}{dt} = y + \lambda + x(x-1) \qquad \frac{dx}{dT} = y + \lambda + x(x-1)$$
$$\frac{dy}{dt} = -x \qquad \frac{dy}{dT} = -\delta x$$

$\delta = 0$



$\delta \neq 0$



Thresholds in multiple timescale systems

$$
\begin{aligned}
\delta \frac{dx}{dt} &= f(x, y, \lambda, \delta) \\
\frac{dy}{dt} &= g(x, y, \lambda, \delta) \\
\frac{d\lambda}{dt} &= \epsilon h(\lambda)
\end{aligned}
$$

$$S := \{f(x, y, \lambda, 0) = 0\}$$

Assumptions:

- 1 fast and 2 slow variables: $0 < \delta << \epsilon < 1$.
- The critical manifold S has a fold F tangent to the fast x direction:
  $$\frac{\partial f}{\partial x} = 0, \ \frac{\partial^2 f}{\partial x^2} \neq 0.$$
- For fixed λ there is single curve of stable states $\tilde{x}(\lambda)$ close to the fold.

Thresholds in multiple timescale systems

$$\delta \frac{dx}{dt} = f(x, y, \lambda, \delta)$$
$$\frac{dy}{dt} = g(x, y, \lambda, \delta)$$
$$\frac{d\lambda}{dt} = \epsilon h(\lambda)$$

$$S := \{f(x, y, \lambda, 0) = 0\}$$

So,

- The critical manifold S is a 2D.
- The fold F seperates S into an attracting part S^a and repelling part S^r.
- The static stable state $\tilde{x}(\lambda)$ is on S^a.

Also, for small $\delta > 0$, S^a and S^r perturb to nearby 2D manifolds S^a_δ and S^r_δ with the same attractivity (Fenichel, 1971).

Look at the system on the critical manifold

$$\begin{aligned}
\delta \frac{dx}{dt} &= f(x, y, \lambda, \delta) \\
\frac{dy}{dt} &= g(x, y, \lambda, \delta) \\
\frac{d\lambda}{dt} &= \epsilon h(\lambda)
\end{aligned}$$

Project onto S:

$$\begin{aligned}
-\frac{\partial f}{\partial x}\Big|_S \frac{dx}{dt} &= \left(g\frac{\partial f}{\partial y} + h\frac{\partial f}{\partial \lambda} \right)\Big|_S \\
\frac{d\lambda}{dt} &= \epsilon h|_S
\end{aligned}$$

$$S := \{f(x, y, \lambda, 0) = 0\}$$

Look at the system on the critical manifold

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\frac{d\lambda}{dt} &= \epsilon h|_S
\end{aligned}
$$

Desingularise by $dt = -ds(\partial f/\partial x)|_S$:

$$
\begin{aligned}
\frac{dx}{ds} &= \left(g\frac{\partial f}{\partial y} + h\frac{\partial f}{\partial \lambda} \right)\bigg|_S \\
\frac{d\lambda}{ds} &= -\epsilon \left(h\frac{\partial f}{\partial x} \right)\bigg|_S
\end{aligned}
$$

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\delta\frac{dx}{dt} &= f(x,y,\lambda,\delta)\\
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-\frac{\partial f}{\partial x}\bigg|_S \frac{dx}{dt} &= \left(g\frac{\partial f}{\partial y} + h\frac{\partial f}{\partial \lambda}\right)\bigg|_S\\
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\frac{d\lambda}{ds} &= -\epsilon\left(h\frac{\partial f}{\partial x}\right)\bigg|_S
\end{aligned}
$$

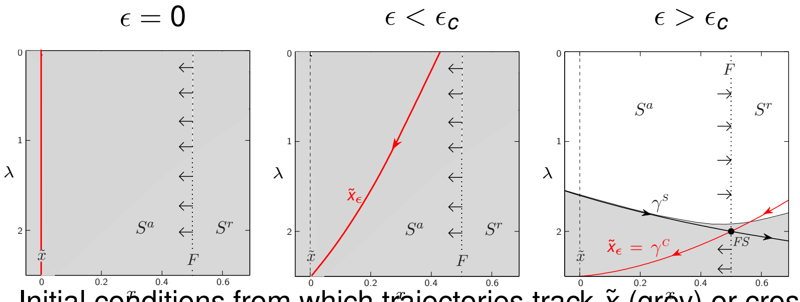
Folded singlarities:

$$\left(g\tfrac{\partial f}{\partial y} + h\tfrac{\partial f}{\partial \lambda}\right)\bigg|_S = 0$$

$$-\tfrac{\partial f}{\partial x}\bigg|_S = 0.$$

Existence of critical rates

There is a critical rate if there is a folded singularity.



Initial conditions from which trajectories track $\tilde{x}$ (grey) or cross F and destabilise (white).

Existence of critical rates

There is a critical rate if there is a folded singularity.

There is a critical rate ϵ_c if for some point (x, y, λ) at time t both

$$\left(g\frac{\partial f}{\partial y} + h\frac{\partial f}{\partial \lambda} \right)\bigg|_F = 0$$

and

$$\frac{d}{d\epsilon}\left(g\frac{\partial f}{\partial y} + h\frac{\partial f}{\partial \lambda} \right)\bigg|_F \neq 0.$$

Non-obvious thresholds

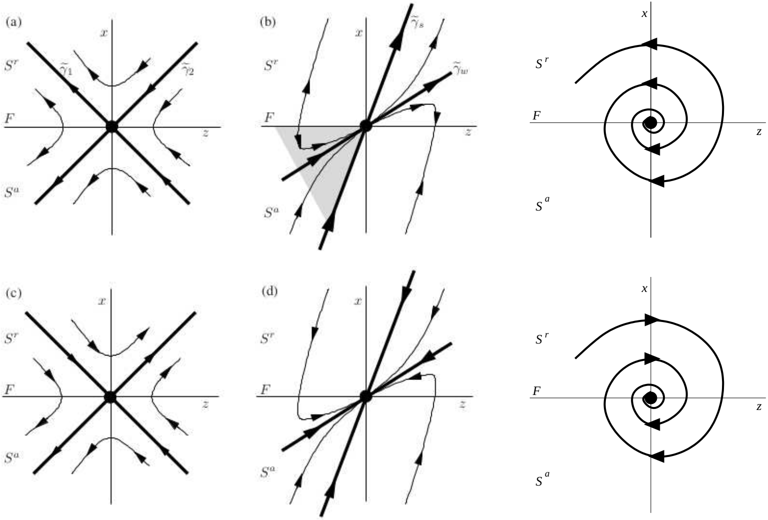
There is a critical rate if there is a folded singularity.

There maybe a threshold seperating the initial conditions from which trajectories track $\tilde{x}$ (grey), and destabilise (white).

Classification of folded singlarities

In the desingularised system time is reverse on S^r.

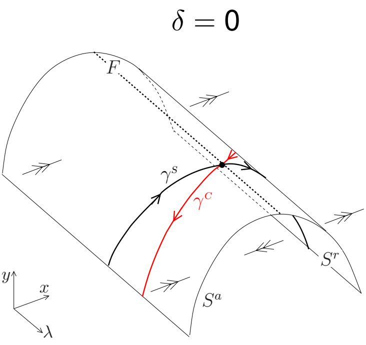
Canards γ go from S^a to S^r along eigendirections.



Desroches et al, 2012

Along the fold F when $\delta \neq 0$

Canard γ_δ^S persists as a transverse (robust) intersection of S_δ^a and S_δ^r, (Szmolyan, 2001). Canards act as seperatrices.

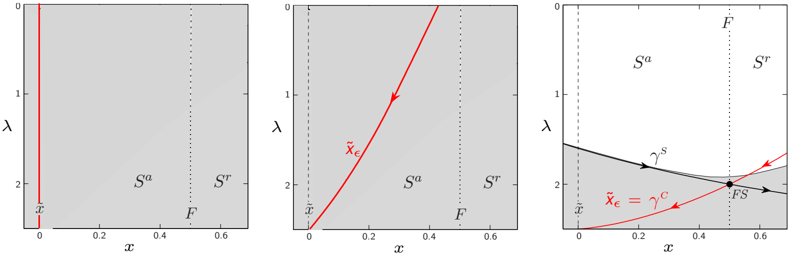


Simple threshold with an isolated folded saddle

$$\delta \frac{dx}{dt} = y + \lambda + x(x-1); \qquad \frac{dy}{dt} = -x; \qquad \frac{d\lambda}{dt} = \epsilon(\lambda_{\max} - \lambda).$$

$$\epsilon_c = \frac{1}{2\lambda_{\max}}.$$

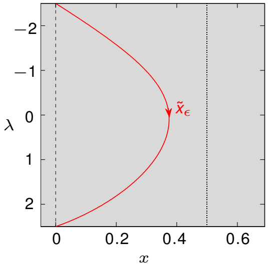
$\epsilon = 0$ $\qquad$ $\epsilon < \epsilon_c$ $\qquad$ $\epsilon > \epsilon_c$

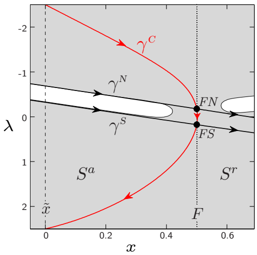


Non-obvious threshold γ_δ^S seperates initial conditions that track $\tilde{x}$ (grey) or destabilise (white).

Complicated threshold after a folded saddle-node bifurcation

$$\delta \frac{dx}{dt} = y + \lambda + x(x-1); \qquad \frac{dy}{dt} = -x; \qquad \frac{d\lambda}{dt} = \epsilon(\lambda_{\max}^2 - \lambda^2)/\lambda_{\max}.$$

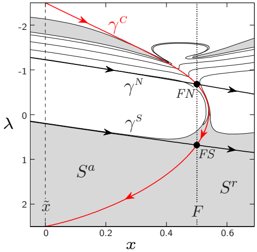


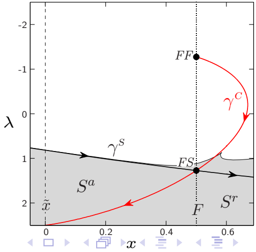


$$\epsilon_c = \frac{1}{2\lambda_{\max}}.$$

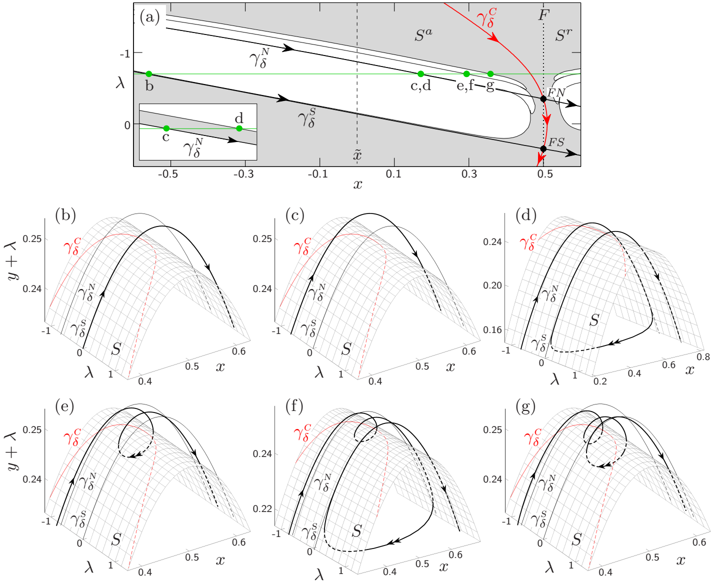
Panel 1: $\epsilon < \epsilon_c$

Panels 2-4: $\epsilon > \epsilon_c$



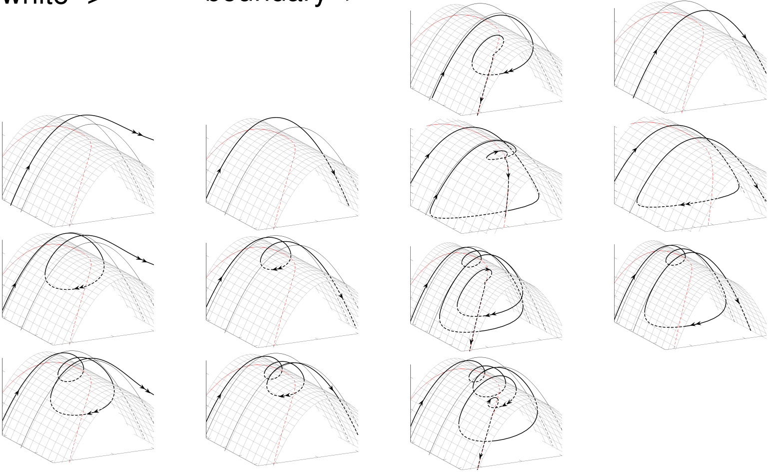


Composite canards form the boundary



Composite canards form the boundary

white -> boundary -> grey -> boundary



Conclusions

Rate-induced bifurcations are of key importance in many real-world system. We seek to identify critical rates and instability thresholds in a range of systems.

Obvious rate-induced bifurcations: find a hetroclinic connection.

Non-obvious rate-induced bifurcations: use geometric singular perturbation theory, find folded singularities and canards:

- Existence results for critical rates and instability thresholds.
- Find a new complicated banded threshold.
- Identify the structure with new composite canards.

Bibliography

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