Network-driven synchronisation of phase-coupled oscillators

Helmut Schmidt

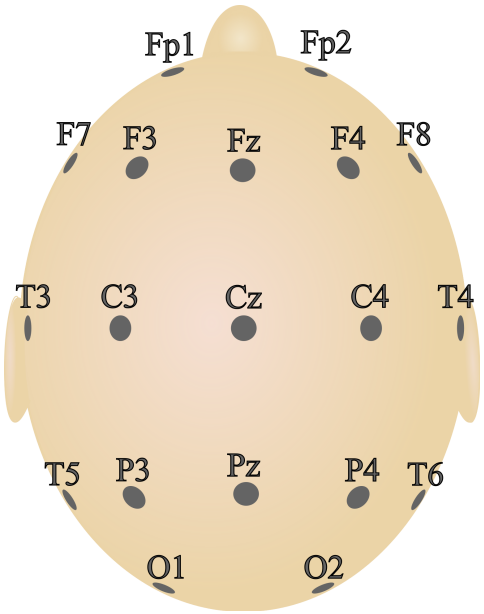
University of Exeter
CEMPS

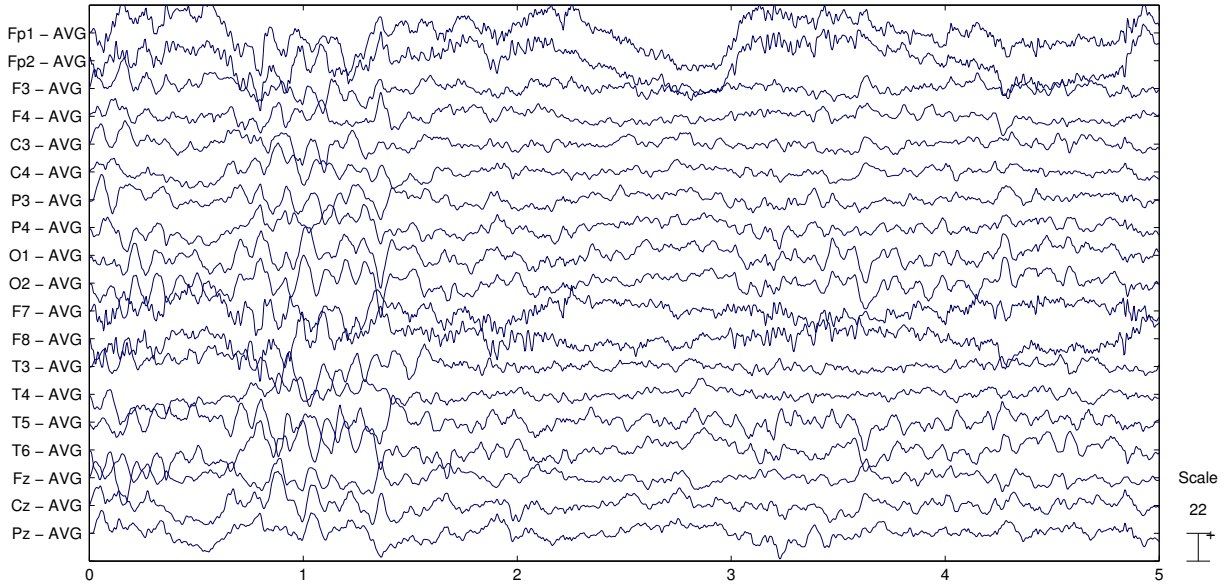
Exeter, 4th Feb 2014

Outline

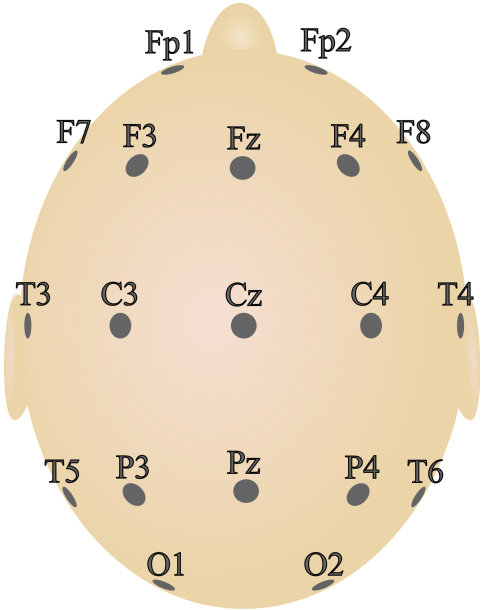
- Motivation
- The 'Classical' Kuramoto Model
- Networks of Kuramoto Models
- Application to EEG Data

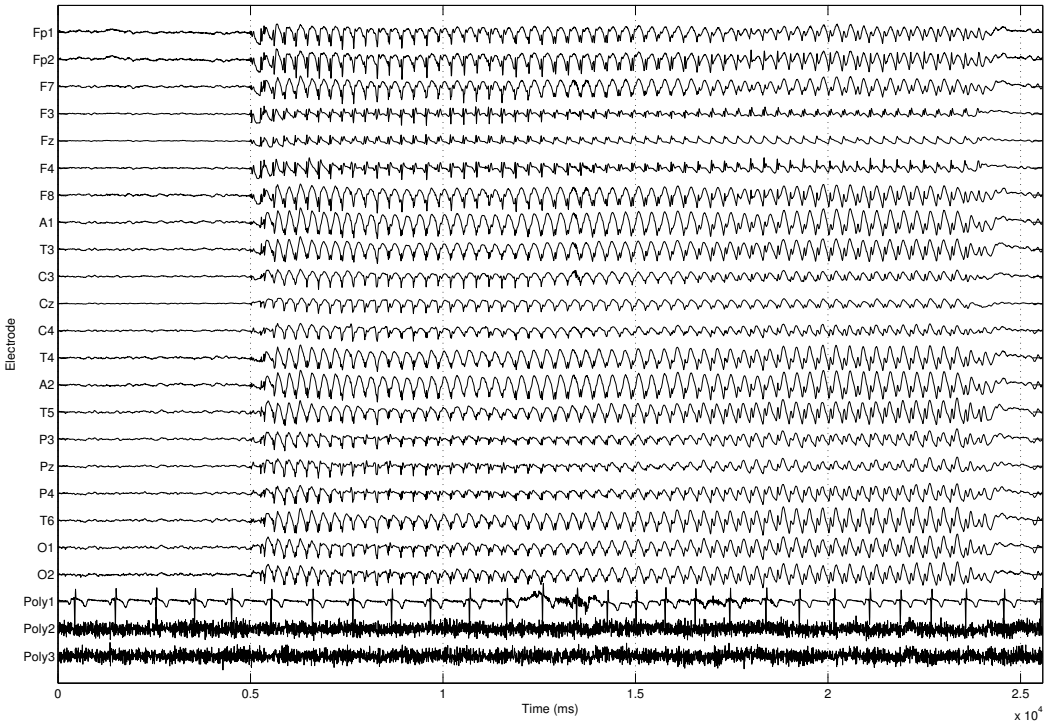
A 'normal' EEG


Fp1
Fp2
F7
F3
Fz
F4
F8
T3
C3
Cz
C4
T4
T5
P3
Pz
P4
T6
O1
O2

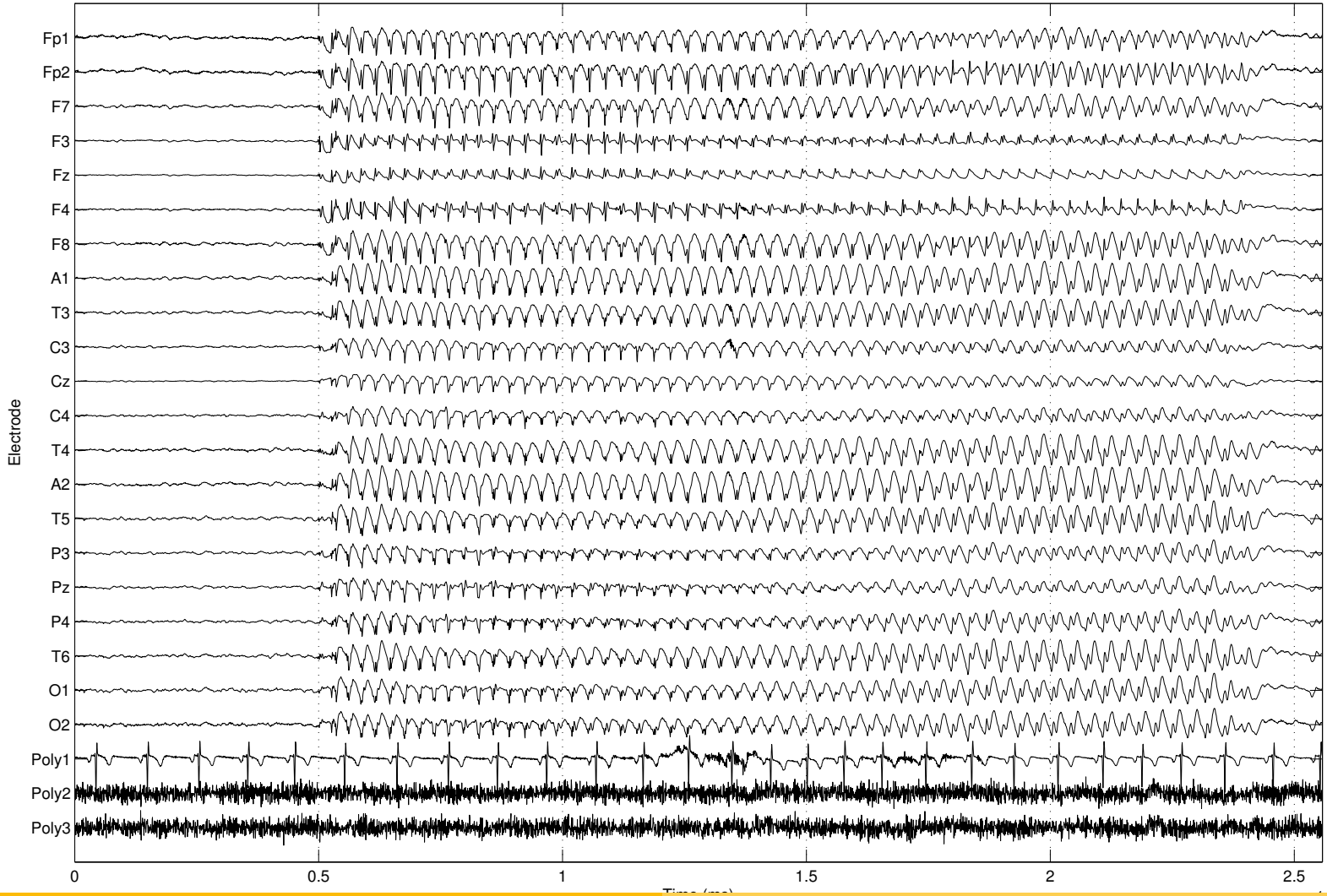

Fp1 – AVG
Fp2 – AVG
F3 – AVG
F4 – AVG
C3 – AVG
C4 – AVG
P3 – AVG
P4 – AVG
O1 – AVG
O2 – AVG
F7 – AVG
F8 – AVG
T3 – AVG
T4 – AVG
T5 – AVG
T6 – AVG
Fz – AVG
Cz – AVG
Pz – AVG
0
1
2
3
4
5
Scale
22

EEG recording of a generalised seizure

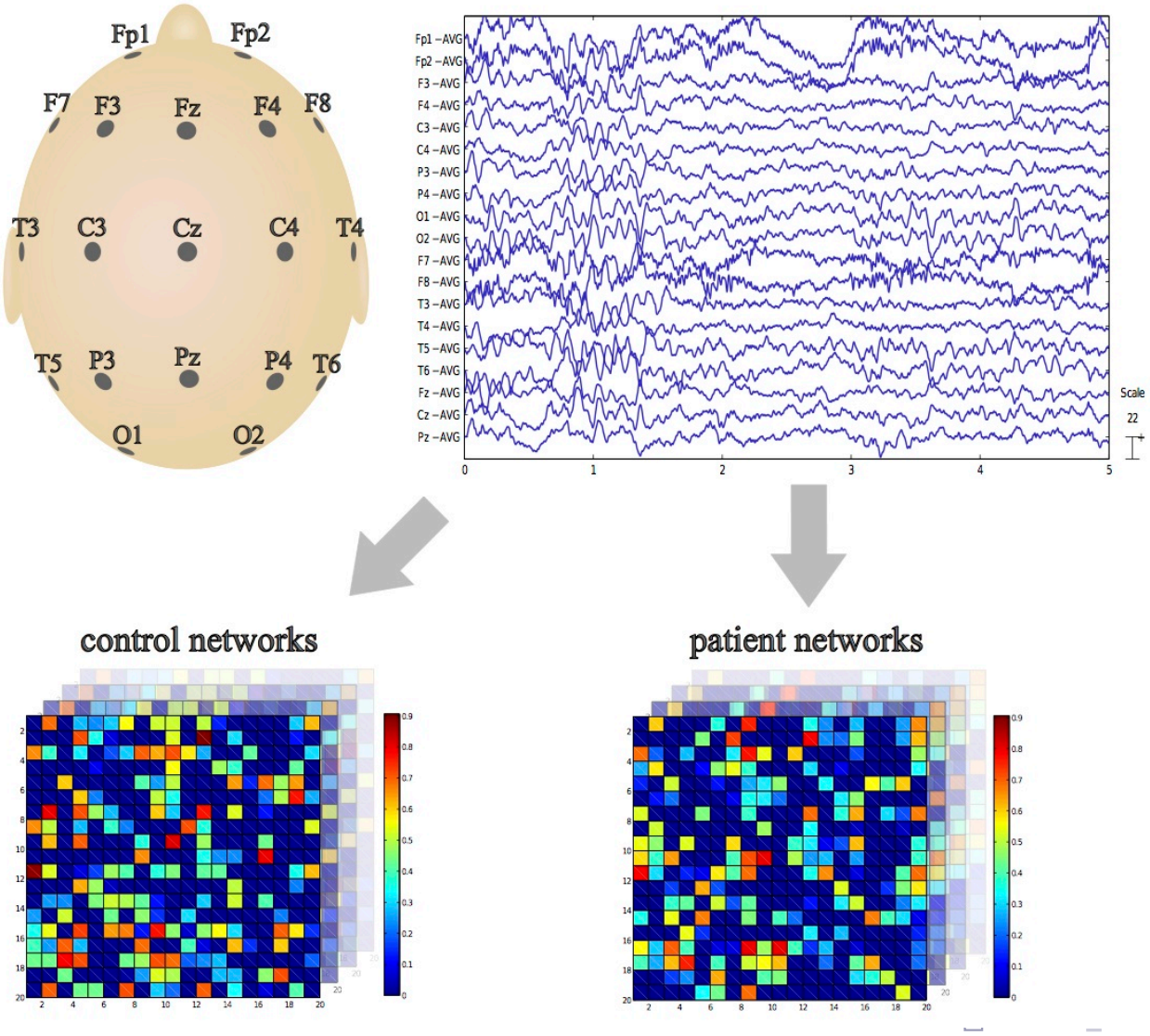




EEG recording of a generalised seizure



Epilepsy - A Network Problem!



Ubiquitousness of the Kuramoto Model

frontiers in NEUROINFORMATICS

ORIGINAL RESEARCH ARTICLE
published: 15 July 2011
doi: 10.3389/fninf.2011.00006

On the influence of amplitude on the connectivity between phases

Andreas Daffertshofer* and Bernadette C. M. van Wijk

Research Institute MOVE, VU University Amsterdam, Amsterdam, Netherlands

Edited by:
Olaf Sporns, Indiana University, USA

Reviewed by:
Joana R. B. Cabral, Universitat Pompeu Fabra, Spain
Juergen Kurths, Humboldt Universität, Germany

****Correspondence:***
Andreas Daffertshofer, Research Institute MOVE, VU University Amsterdam, Van der Boechorststraat 9, 1081 BT Amsterdam, Netherlands.
e-mail: a.daffertshofer@vu.nl

In recent studies, functional connectivities have been reported to display characteristics of complex networks that have been suggested to concur with those of the underlying structural, i.e., anatomical, networks. Do functional networks always agree with structural ones? In all generality, this question can be answered with "no": for instance, a fully synchronized state would imply isotropic homogeneous functional connections irrespective of the "real" underlying structure. A proper inference of structure from function and *vice versa* requires more than a sole focus on phase synchronization. We show that functional connectivity critically depends on amplitude variations, which implies that, in general, phase patterns should be analyzed in conjunction with the corresponding amplitude. We discuss this issue by comparing the phase synchronization patterns of interconnected Wilson–Cowan models vis-à-vis Kuramoto networks of phase oscillators. For the interconnected Wilson–Cowan models we derive analytically how connectivity between phases explicitly depends on the generating oscillators' amplitudes.

$$\frac{d}{dt}E_k = -E_k + S\left[a_E\left(c_{EE}E_k - c_{IE}I_k - \theta_E + P_k + \frac{\eta}{N}\sum_{l=1}^{N} C_{kl}E_l\right)\right]$$

$$\frac{d}{dt}I_k = -I_k + S\left[a_I\left(c_{EI}E_k - c_{II}I_k - \theta_I\right)\right]$$

$$\Longrightarrow$$

$$\frac{d}{dt}\varphi_k = \omega_k + \frac{\eta}{2N}\sum_{l=1}^{N} C_{kl}a_E S'\left[\chi_{E,k}^{(0)}\right]\frac{R_l}{R_k}\sin\left(\varphi_l - \varphi_k\right)$$
$$+ \frac{\eta}{16N}\sum_{l=1}^{N} C_{kl}a_E^3 S'''\left[\chi_{E,k}^{(0)}\right]R_k R_l\left(\left(c_{EE}^2 + 3c_{IE}^2\right)\sin\left(\varphi_l - \varphi_k\right)\right.$$
$$\left. + 2c_{EE}c_{IE}\cos\left(\varphi_l - \varphi_k\right)\right)$$

The Kuramoto Model

- The model:

$$\dot{\theta}_j = \omega_j + \frac{K}{N}\sum_{n=1}^{N}\sin(\theta_n - \theta_j), \qquad j = \{1, \ldots, N\}$$

- The order parameter r defines the degree of 'synchronicity':

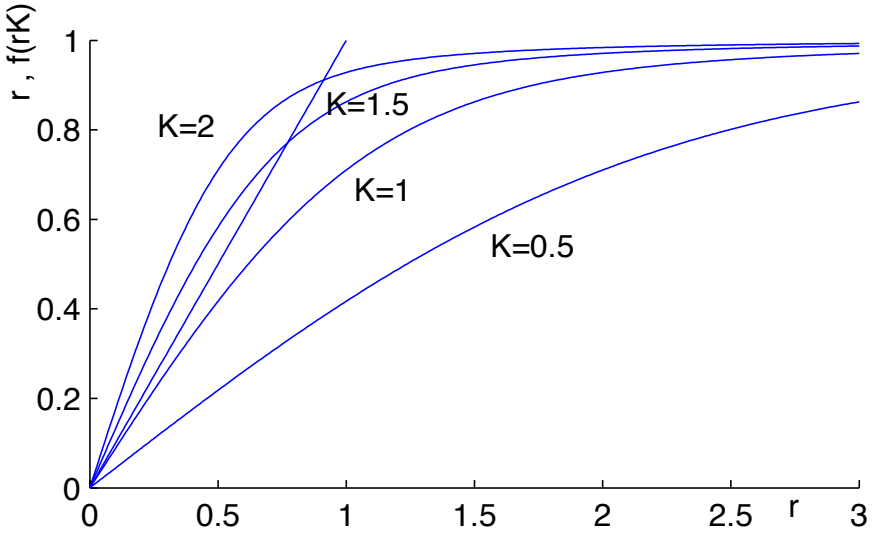
$$r\mathrm{e}^{\mathrm{i}\psi} = \frac{1}{N}\sum_{n=1}^{N}\mathrm{e}^{\mathrm{i}\theta_n} \quad \Longrightarrow \quad \dot{\theta}_j = \omega_j + rK\sin(\psi - \theta_j),$$

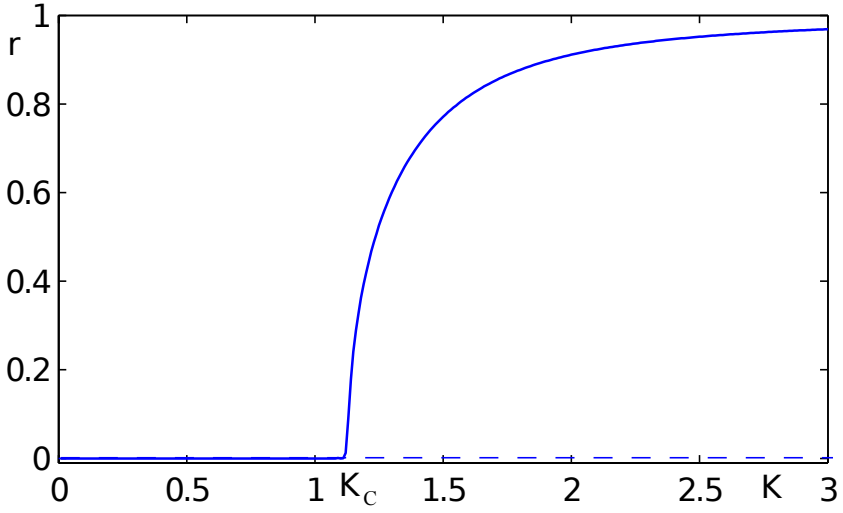
- Thermodynamic limit ($N \to \infty$), $g(\omega) = \frac{1}{\sqrt{\pi}}\mathrm{e}^{-(\Omega-\omega)^2}$:

$$r = \sqrt{\pi}\frac{rK}{2}\mathrm{e}^{-(rK)^2/2}\left(\mathrm{I}_0\left(\frac{(rK)^2}{2}\right) + \mathrm{I}_1\left(\frac{(rK)^2}{2}\right)\right) = F(rK)$$

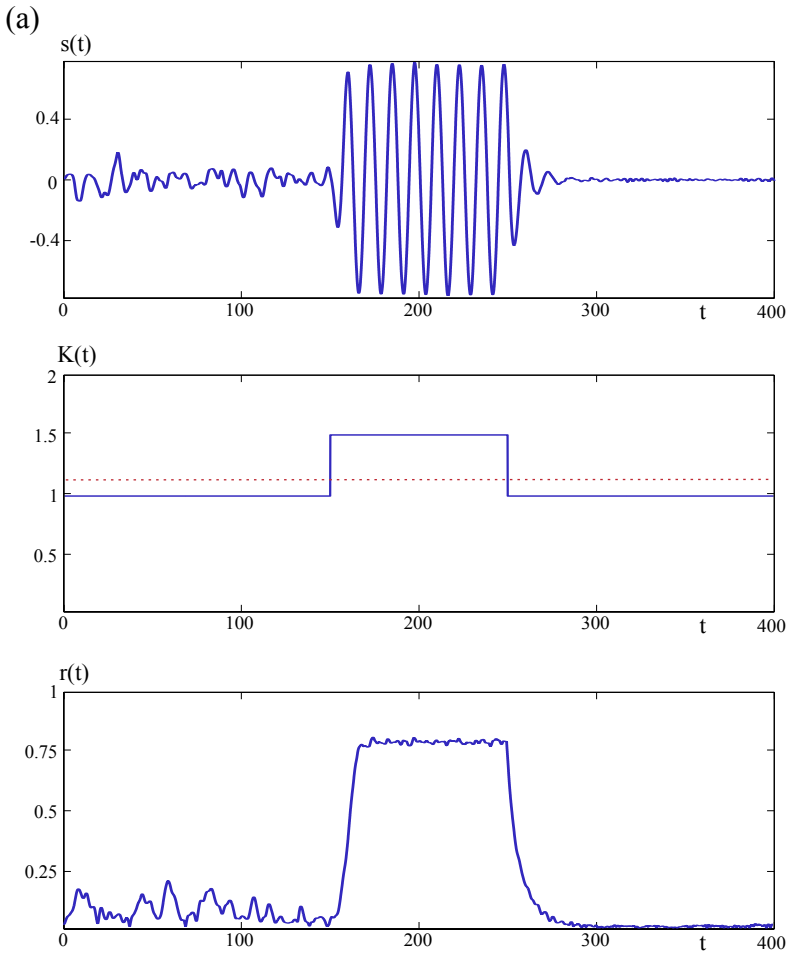
The Kuramoto Model

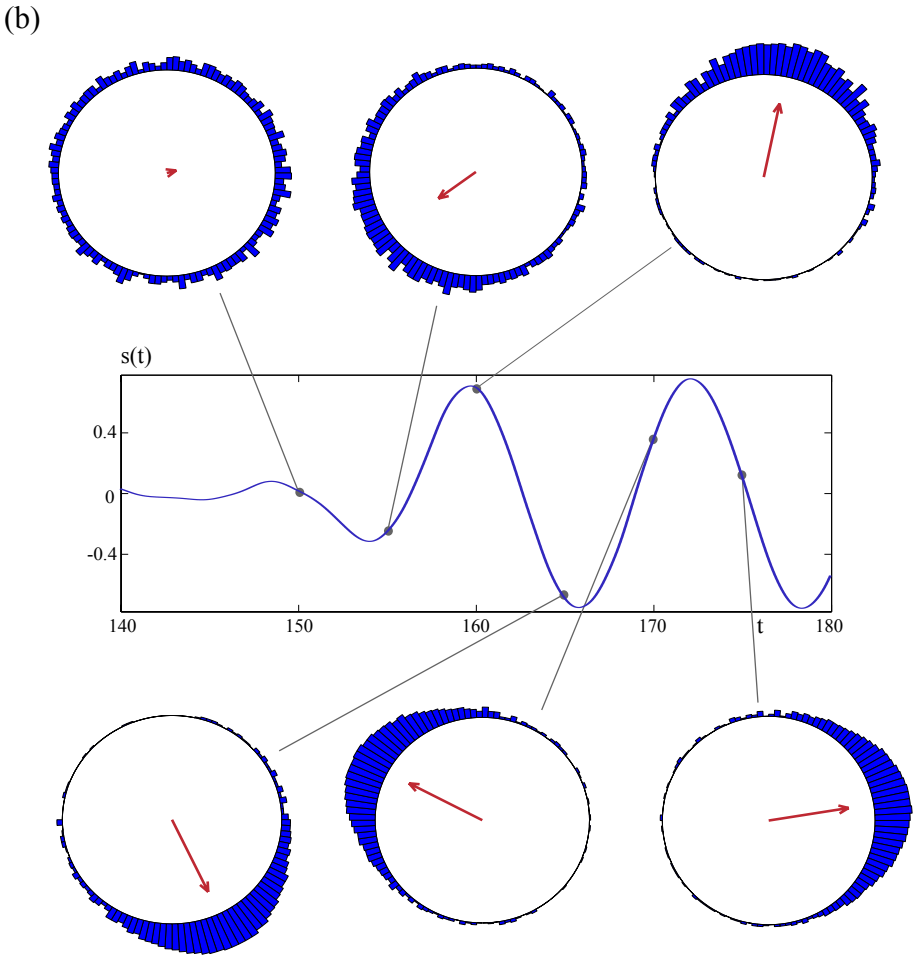
- Synchronisation sets in at critical coupling strength K_c:



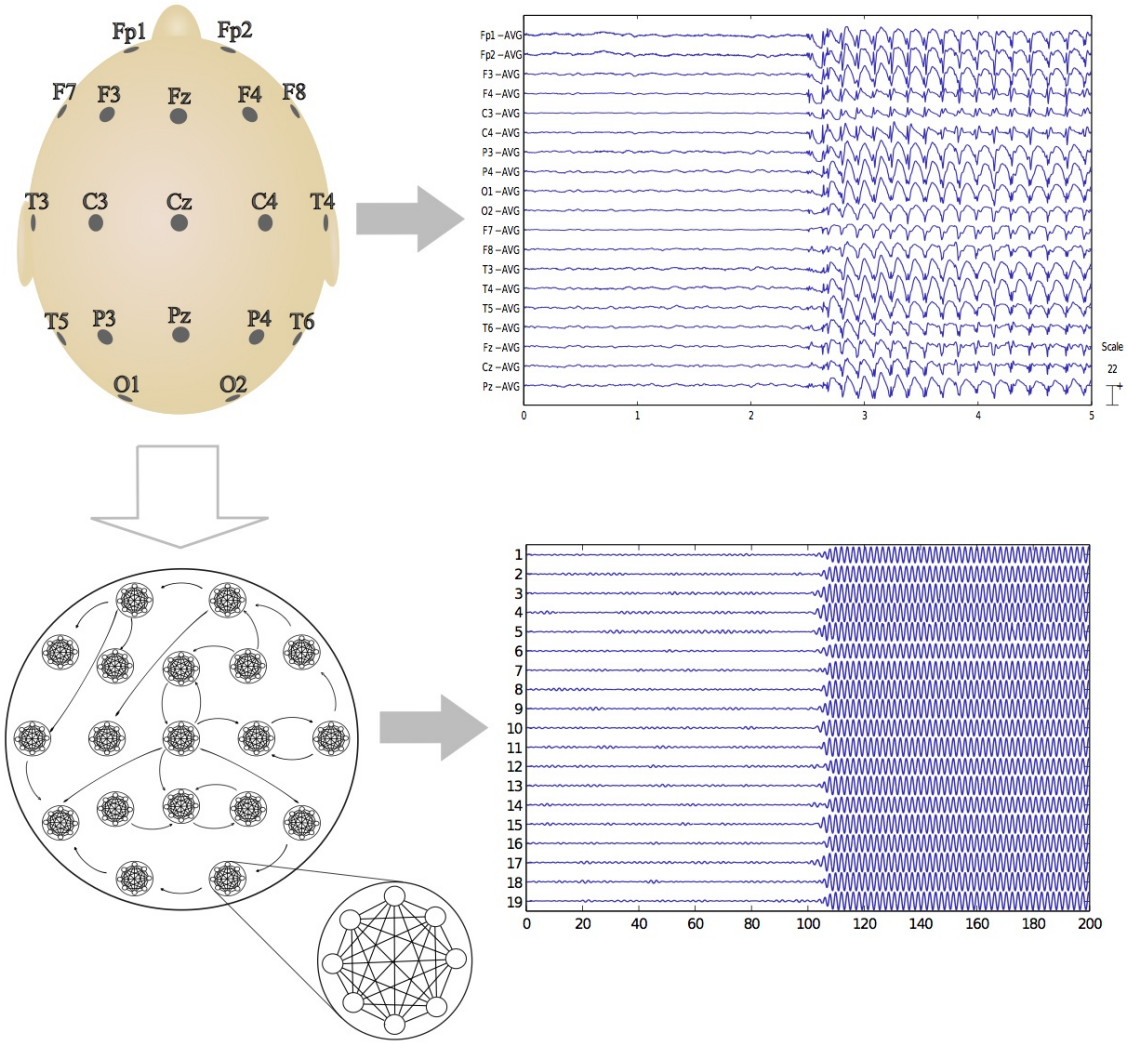


The Kuramoto model





Directed Networks of 'Kuramoto Populations':



Directed Networks of 'Kuramoto Populations':

- The full model with adjacency matrix $\hat{A}$:

$$\dot{\theta}_j^p = \omega_j^p + \frac{K_p}{N}\sum_{n=1}^{N}\sin(\theta_n^p - \theta_j^p) + \sum_{s=1}^{P}\frac{A_{s,p}}{N}\sum_{n=1}^{N}\sin(\theta_n^s - \theta_j^p),$$

- define local order parameters:

$$r_p e^{\mathrm{i}\psi_p} = \frac{1}{N}\sum_{n=1}^{N} e^{\mathrm{i}\theta_n^p},$$

$\Longrightarrow$

$$\dot{\theta}_j^p = \omega_j^p + r_p K_p \sin(\psi_p - \theta_j^p) + \sum_{s=1}^{P} r_s A_{s,p}\sin(\psi_s - \theta_j^p).$$

Directed Networks of 'Kuramoto Populations':

- under the assumption that $\psi_p = \psi_s \, \forall \, p, s$ we obtain

$$r_p = F\left(K_p r_p + \sum_{s=1}^{P} A_{s,p} r_s\right), \qquad p = \{1, \ldots, P\}.$$

- linearisation about $(r_1, \ldots, r_P) = (0, \ldots, 0)$ yields

$$\left(\frac{K_p}{K_c} - 1\right) r_p + \frac{1}{K_c} \sum_{m=1}^{P} A_{s,p} r_s = 0.$$

- nontrivial solutions exist if

$$\det\left(\frac{\hat{A}}{K_c} + \hat{\mathcal{K}}\right) = 0, \quad \mathcal{K}_{p,p} = \frac{K_p}{K_c} - 1, \quad \mathcal{K}_{s,p} = 0 \text{ if } s \neq p$$

Directed Networks of 'Kuramoto Populations':

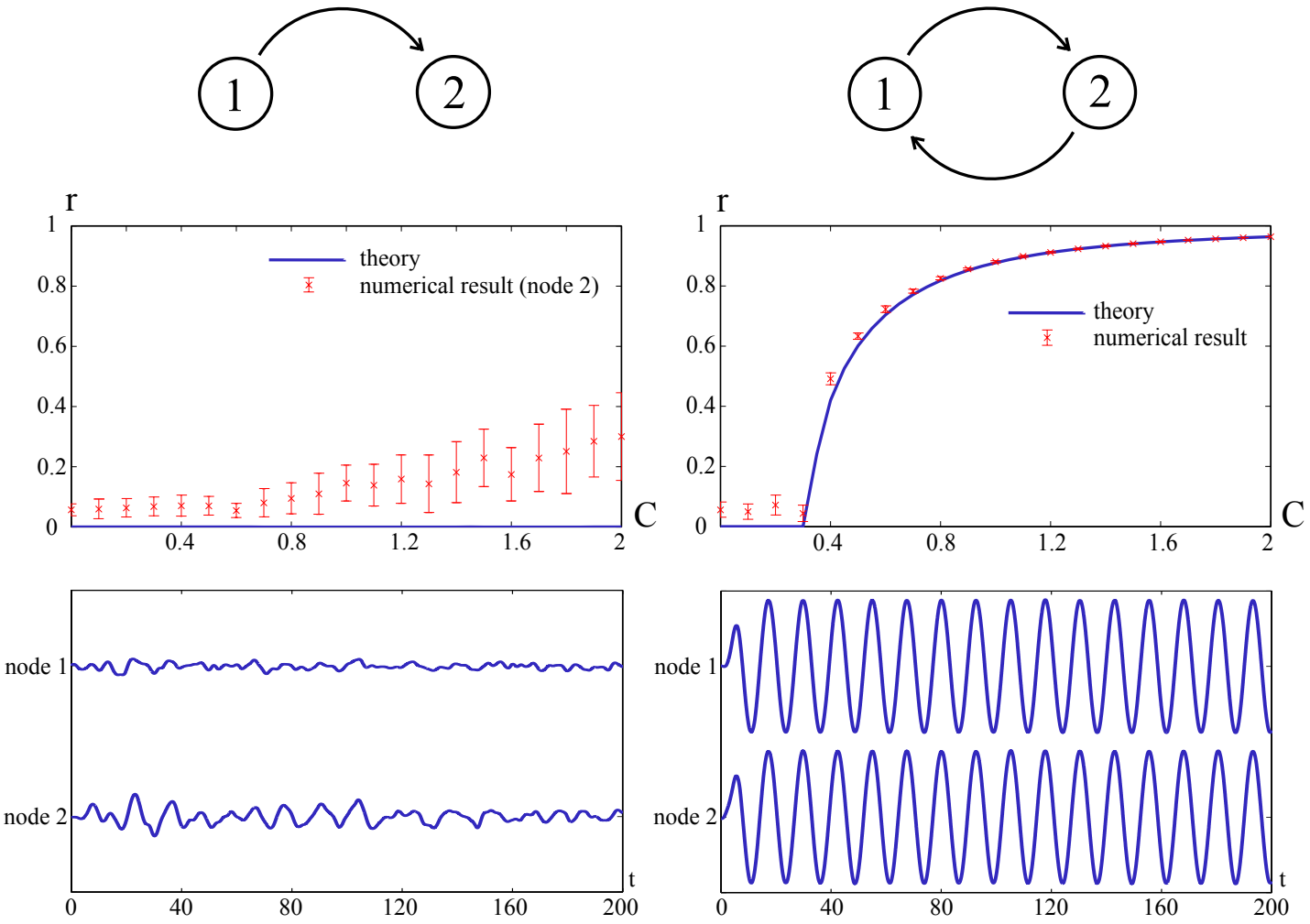
- rewriting $\hat{A} = C\hat{\rho}$ yields a critical value for the global coupling parameter C:

$$\det\left(\frac{C_c}{K_c}\hat{\rho} + \hat{\mathcal{K}}\right) = 0.$$

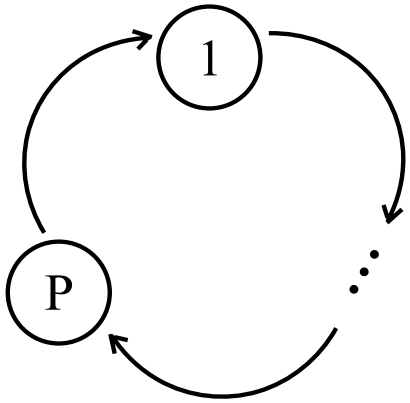
- if $\hat{\rho}$ is triangular,

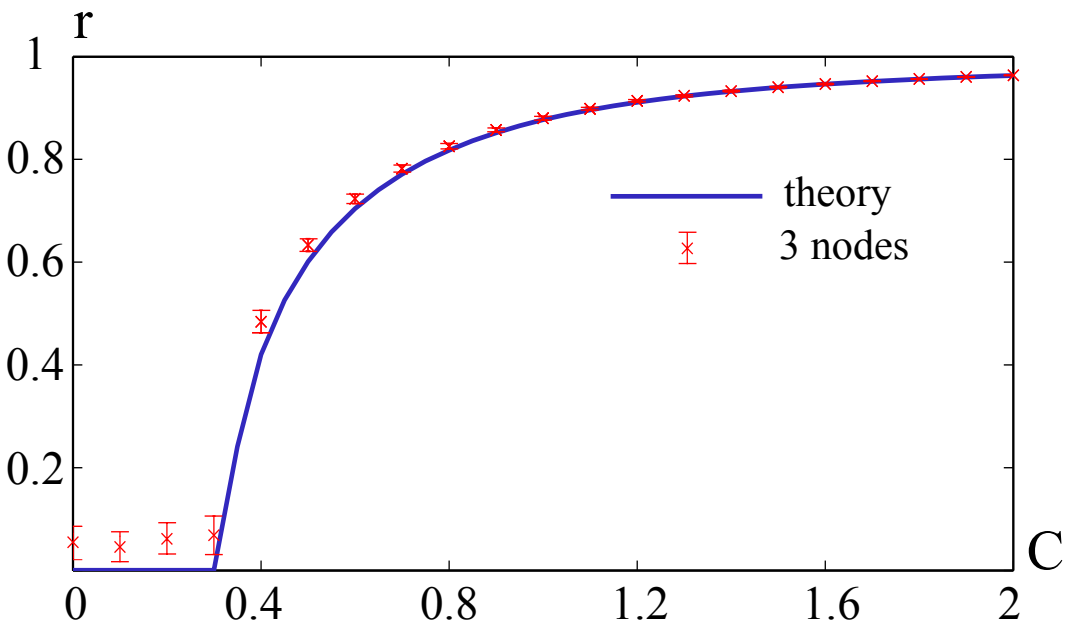
$$\det\left(\frac{C_c}{K_c}\hat{\rho} + \hat{\mathcal{K}}\right) = \det\left(\hat{\mathcal{K}}\right) \neq 0$$

2 populations

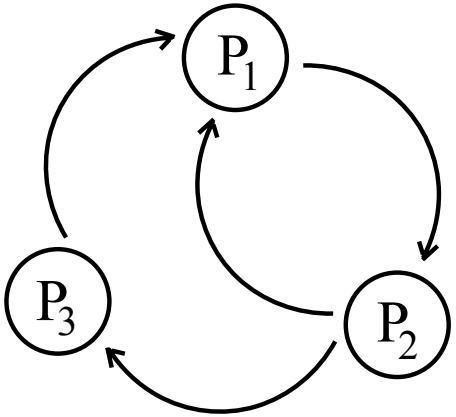


cycles

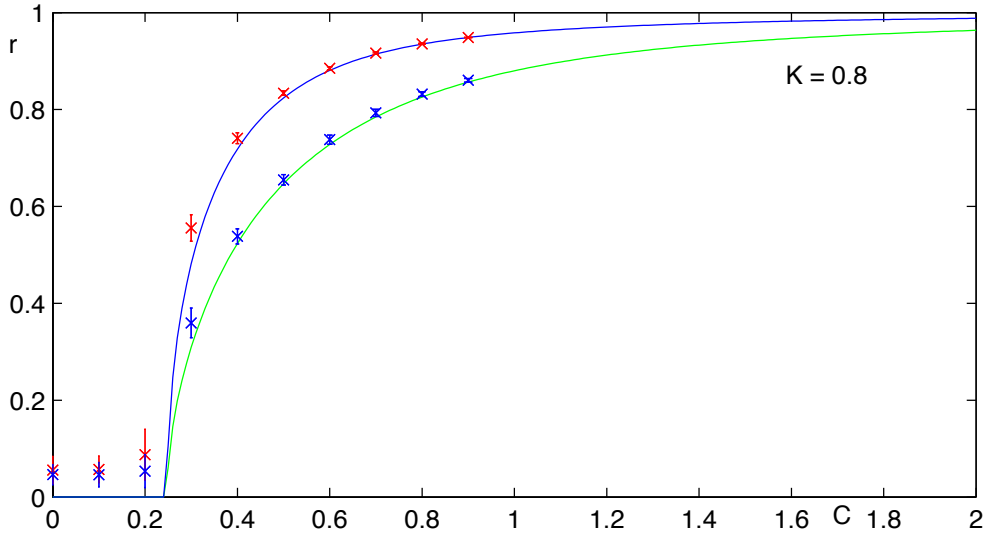




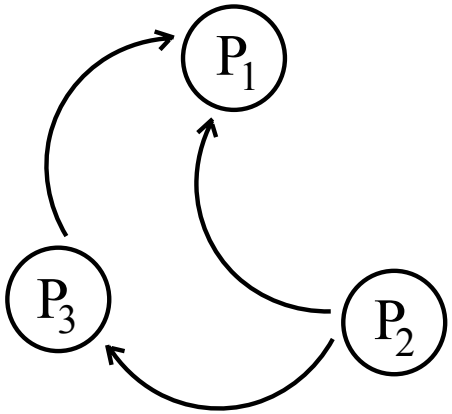
a loop of 3 populations plus one extra connection



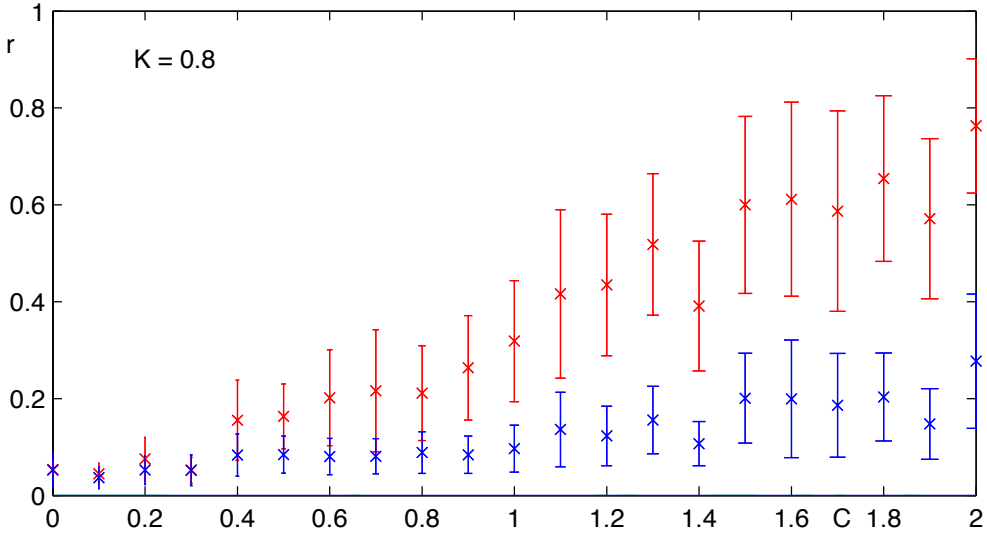
$$r_1 = F(r_1 K + r_2 C + r_3 C)$$
$$r_2 = F(r_2 K + r_1 C)$$
$$r_3 = F(r_3 K + r_2 C)$$



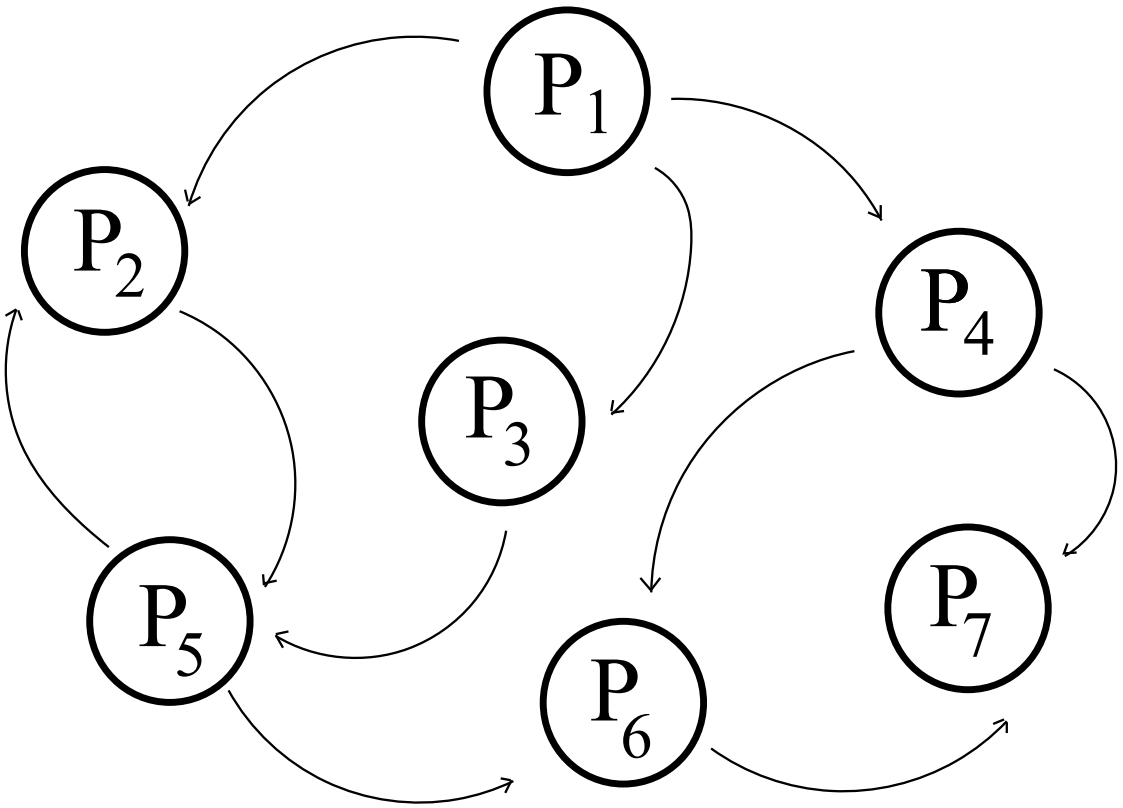
a hierarchical network of 3:



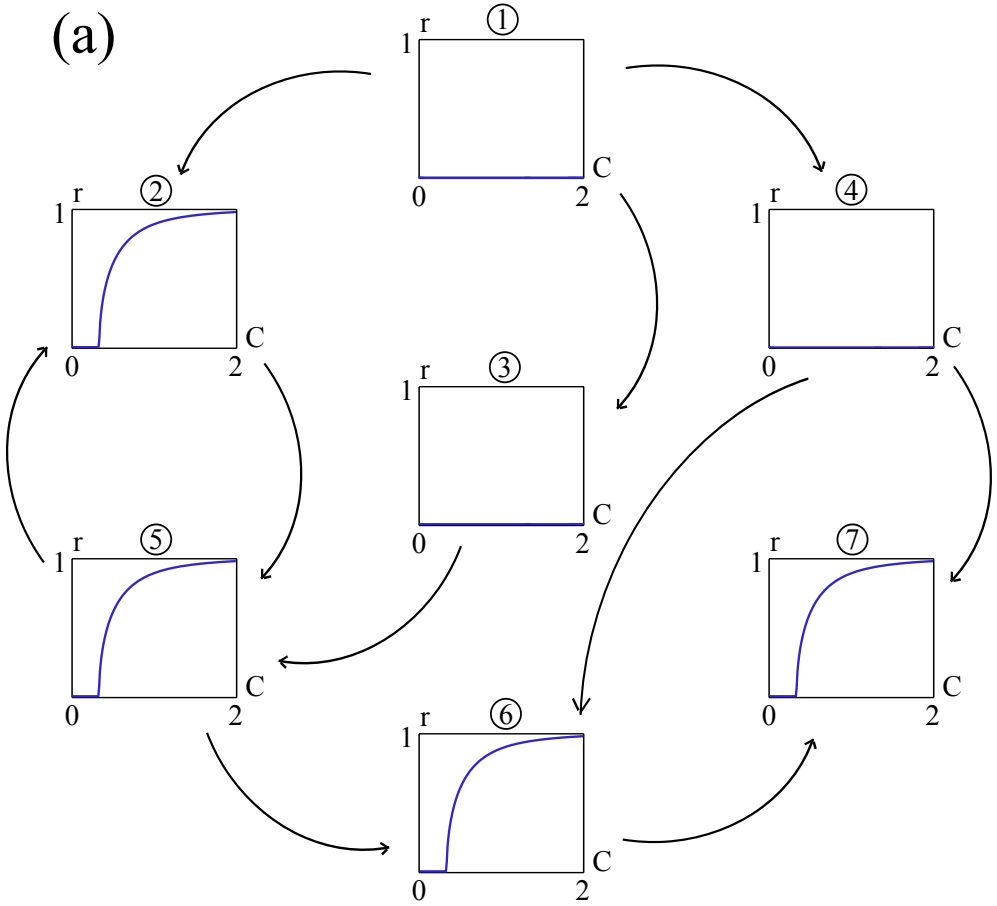
$$r_1 = F(r_1 K + r_2 C + r_3 C)$$
$$r_2 = F(r_2 K)$$
$$r_3 = F(r_3 K + r_2 C)$$



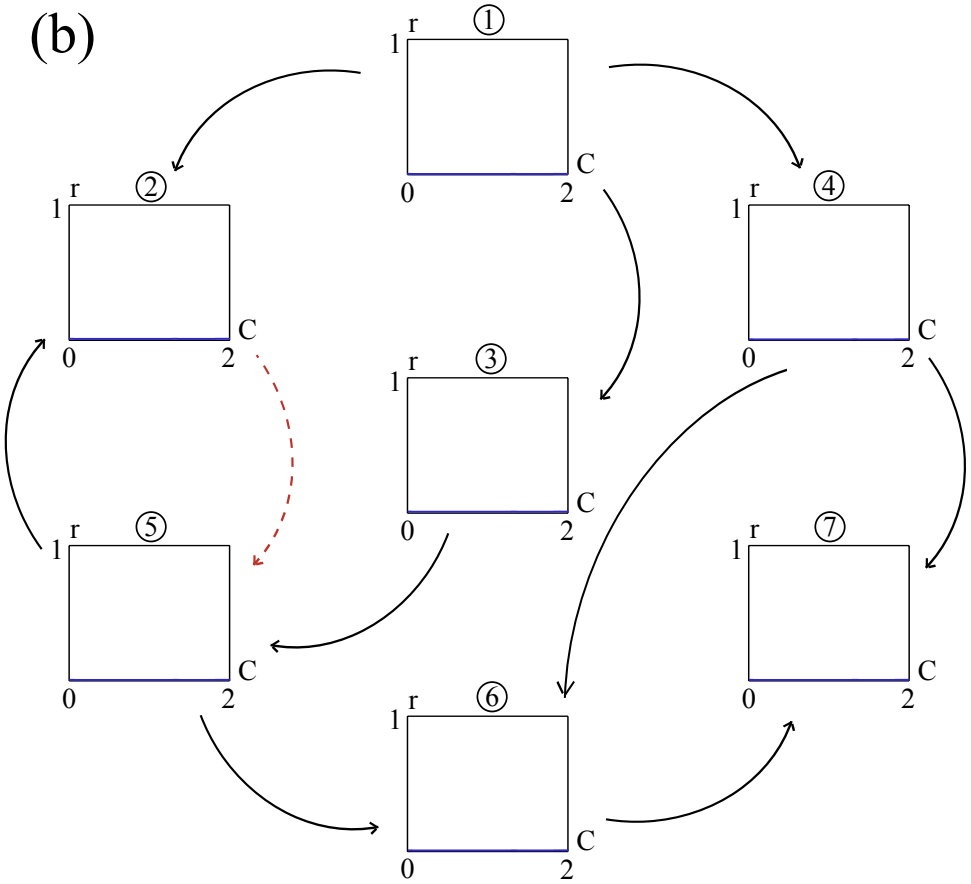
Larger Networks



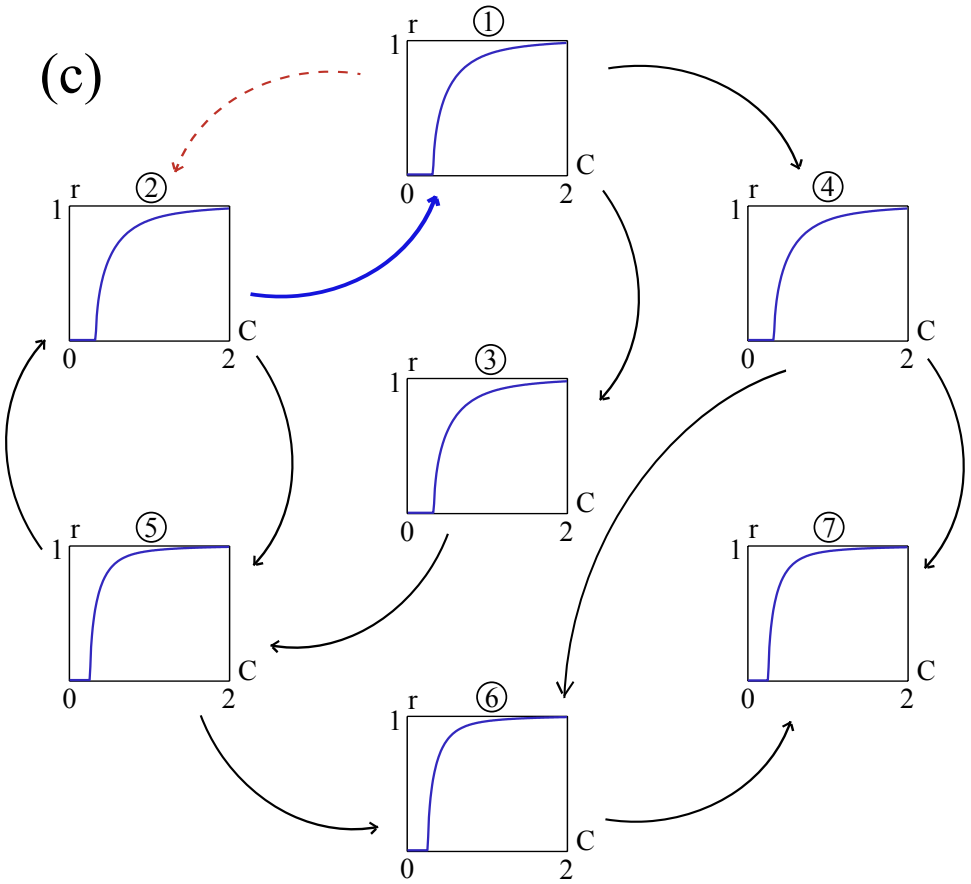
Larger Networks



Larger Networks



Larger Networks



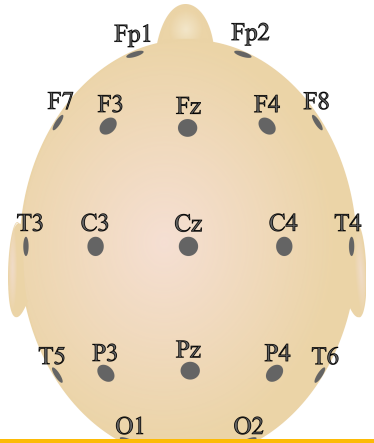
Directed Networks from EEG Data

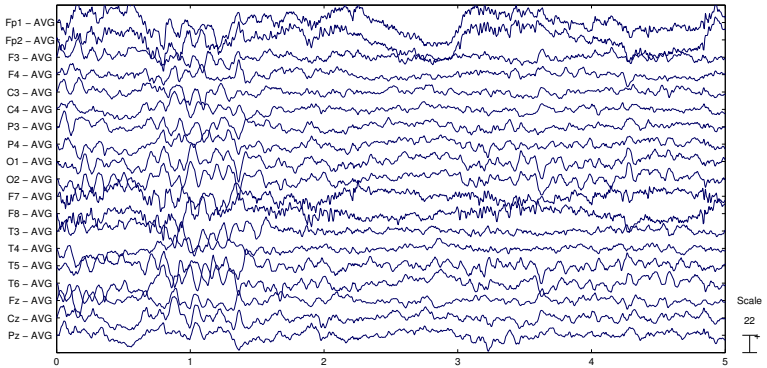
- To compute (delayed) cross-correlation between time-series x_i and x_j we follow Bialonski & Lehnertz (2013):

$$\rho_{i,j} = \max_{\tau} \left\{ \left| \frac{\xi(x_i, x_j)(\tau)}{\sqrt{\xi(x_i, x_i)(0)\xi(x_j, x_j)(0)}} \right| \right\},$$

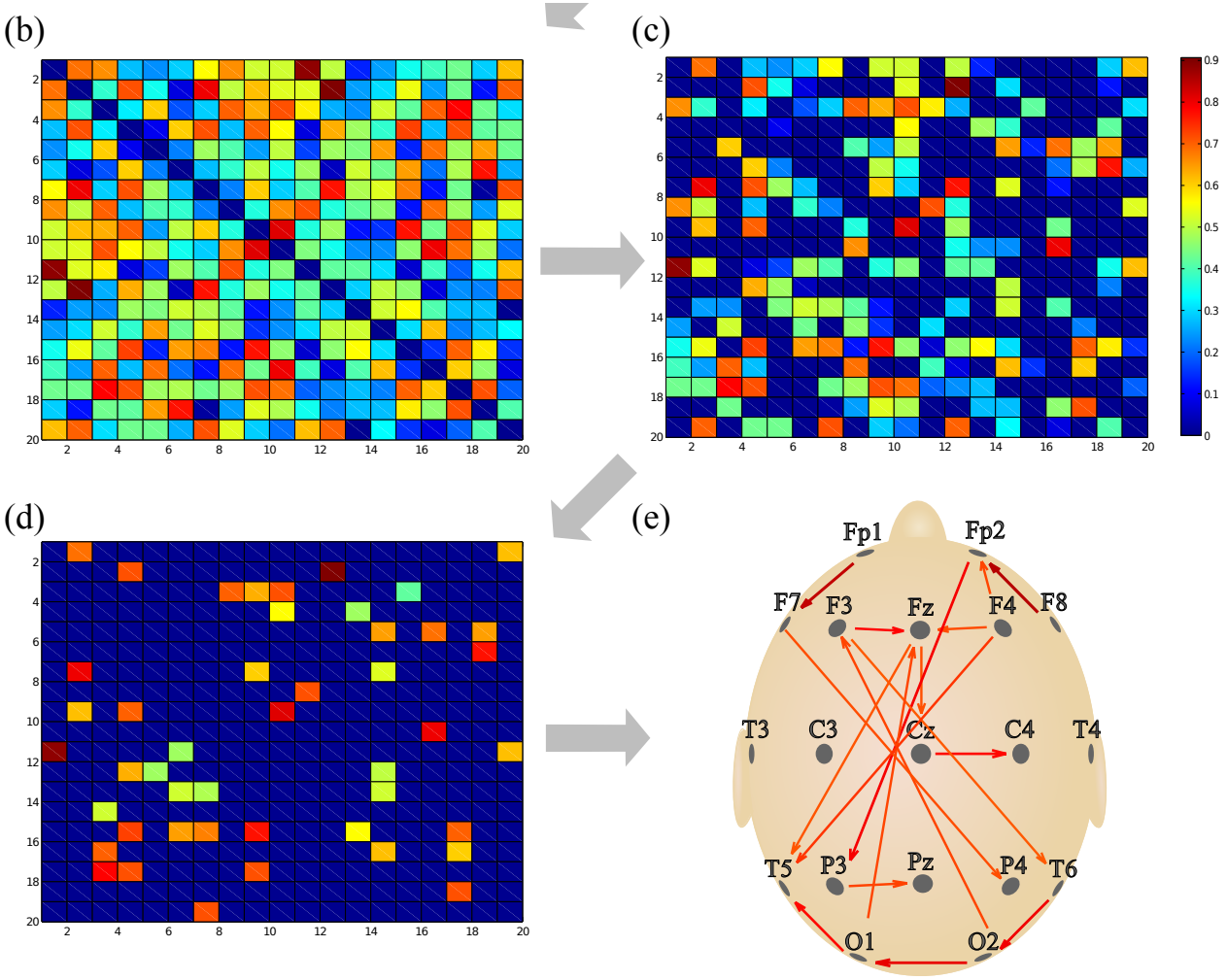
with

$$\xi(x_i, x_j)(\tau) = \begin{cases} \sum_{t=1}^{T-\tau} x_i(t+\tau)x_j(t) & \text{if } \tau \geq 0 \\ \sum_{t=1+\tau}^{T} x_i(t+\tau)x_j(t) & \text{if } \tau < 0 \end{cases}.$$

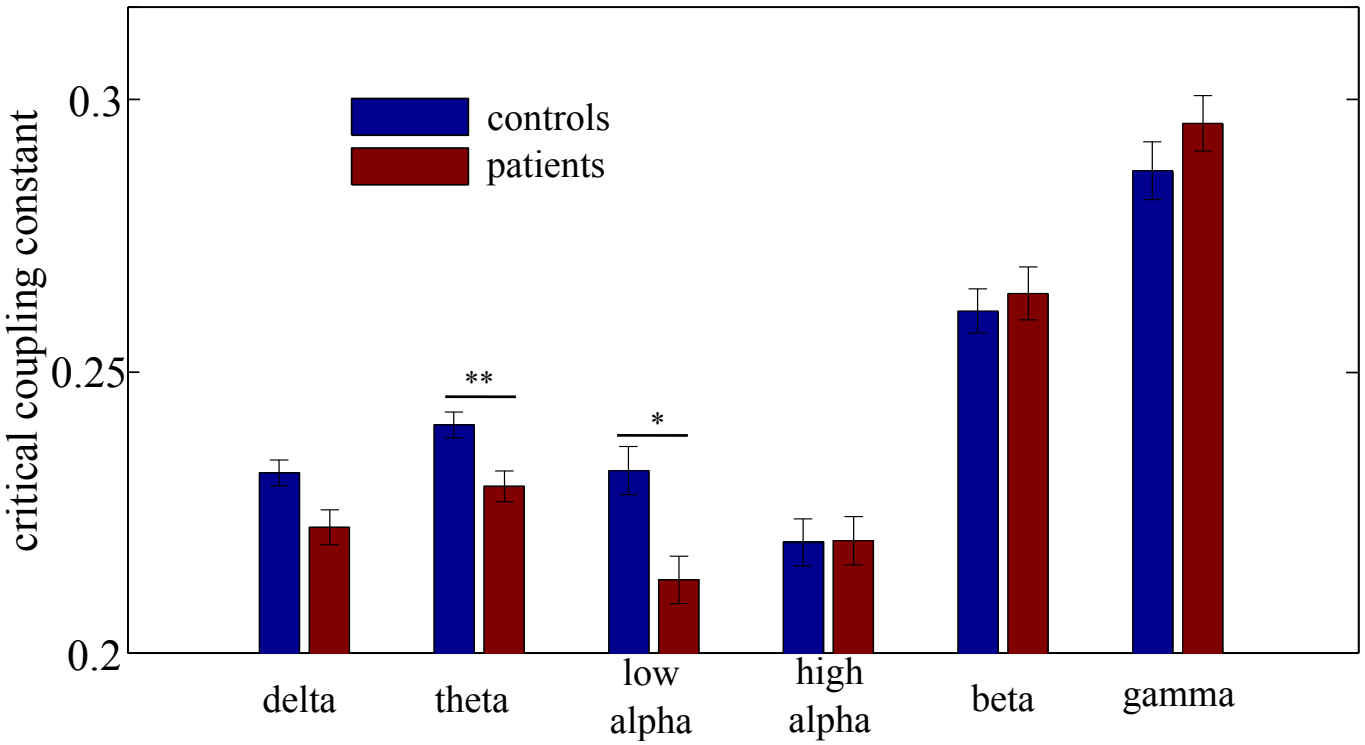




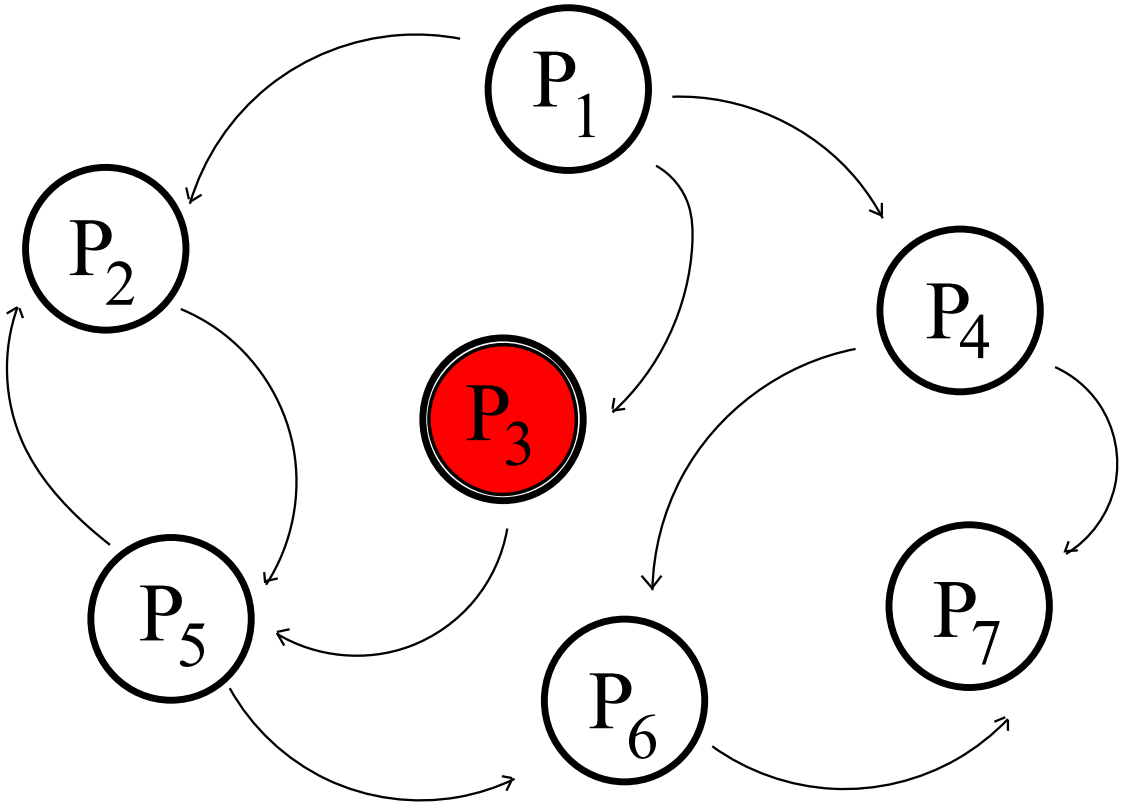
Directed Networks from EEG Data



Critical Coupling Constant C_c



Seizure Onset Zones



Seizure Onset Zones