

Attractors and attracting measures

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Purpose

The purpose of this talk is not just to **explain** some of what I've been thinking about with Pete, but also to **get some feedback and learn**.

→ I have pretty much no experience with smooth ergodic theory.

I will probably be “asking” at least as many questions as I “answer”.

Motivation

Deterministic model of a climate system "without climate change":

- autonomous DE $\dot{x} = F(x)$ on some state space M , ...
 - autonomous since climate parameters are not changing;
 - but there may be different regions of M corresponding to "qualitatively different" climate scenarios, so:
- ... with some given "attractor" $A \subset M$ representing a **stable qualitative state of the climate system**.

E.g. AMOC could be described by an ODE $\dot{x} = F(x)$ with an attractor A_{on} corresponding to the "on" state of the AMOC (while there is another attractor A_{off} corresponding to a qualitatively different stable state of the AMOC, the "off" state).

Motivation

Without “climate change”, the **quantitative state** of the climate is always changing!

⇒ the attractor A is not a single point.

Leads to the question:

“If I observe this deterministic climate system at some ‘random’ time t that has nothing to do with the state of the climate itself, what is the probability distribution for the current quantitative state $x(t)$ of the climate?”

Note:

- This would be a probability distribution supported on A .
- If A is “**chaotic**”, then the answer is unaffected by observations made a sufficiently long time ago.

Axiom A attractors

M – compact Riemannian manifold

m – Lebesgue measure

$(f^t)_{t \in \mathbb{R}}$ – solution flow for an autonomous ODE on M

An **Axiom A attractor** $A \subset M$ is a chaotic attractor with very nice “hyperbolicity” properties.

[Bowen & Ruelle, 1975]¹ Under weak conditions, given an Axiom A attractor A with “basin of attraction”

$$B_A := \{x_0 \in M : d(f^t x_0, A) \rightarrow 0 \text{ as } t \rightarrow \infty\},$$

A is the support of an ergodic invariant probability measure μ_A with the **following two properties**:

¹Analogous result for discrete-time maps in [Ruelle, 1976].

Axiom A attractors

① [BR75, Thm. 5.1] For m -a.e. $x_0 \in B_A$,

$$\frac{1}{t} \int_0^t \delta_{f^s x_0} ds \xrightarrow{\text{weakly}} \mu_A \text{ as } t \rightarrow \infty.$$

→ LHS = law of $x(T)$ for $T \sim \text{Unif}(0, t)$, given $x(0) = x_0$.

Property 1 is **analogous to ergodicity**: a prob. meas. μ is ergodic iff for μ -a.e. $x_0 \in M$,

$$\frac{1}{t} \int_0^t \delta_{f^s x_0} ds \xrightarrow{\text{weakly}} \mu.$$

Heuristic meaning of " m -a.e.": your i.c. has to be "infinitely unlucky" to be a counterexample.

Axiom A attractors

② [BR75, Thm. 5.3] For every p.m. $\nu_0 \ll m$ with $\nu_0(B_A) = 1$,

$$f^t \nu_0 \xrightarrow{\text{weakly}} \mu_A \text{ as } t \rightarrow \infty.$$

→ LHS = law of $x(t)$ for $x(0) \sim \nu_0$.

Property 2 is **analogous to mixing**: a p.m. μ is mixing iff \forall p.m. $\nu_0 \ll \mu$,

$$f^t \nu_0 \xrightarrow[\text{(or strongly)}]{\text{weakly}} \mu.$$

(Remark. mixing \Rightarrow ergodic.)

Heuristic meaning of " $\nu_0 \ll m$ ": the initial distribution ν_0 respects the notion of an "infinitely unlucky" result for $x(0)$.

Axiom A attractors

The analogy between Property 2 and mixing can be understood in terms of **decay of "classical correlations"** vs. **decay of "operational correlations"** (e.g. [Baladi *et al.*, 2002]):

- An invariant p.m. μ is mixing iff $\forall g_1, g_2 \in C_b(M, \mathbb{R})$,

$$\text{Cov}_\mu[g_1, g_2 \circ f^t] = \int g_1(x) g_2(f^t x) \mu(dx) - \int g_1 d\mu \int g_2 d\mu \rightarrow 0.$$

- Property 2 can be re-expressed as: $\forall g_1, g_2 \in C_b(B_A, \mathbb{R})$,

$$\text{OC}(g_1, g_2, t) := \int_{B_A} g_1(x) g_2(f^t x) m(dx) - \int_{B_A} g_1 dm \int g_2 d\mu_A \rightarrow 0.$$

Axiom A attractors

Note: in [BR75], μ_A is constructed in Sec. 3; then Thms. 5.1 and 5.3 are separately proved based on material developed prior to Sec. 5.

Remark:

- Pr. 1 holds in the general setting of [BR75].
 - **Pr. 2 holds under mild extra assumption:** the unstable manifold of each point in A is dense in A .
- Extra assumption **only enters via the fact that it implies μ_A is mixing**.

So [BR75, Thm. 5.3] "really" says: μ_A mixing $\Rightarrow \mu_A$ has Pr. 2.

“Attracting measures”

Property 1: “physical measures”

→ extensively studied in more general settings

Property 2: ???

→ I will call such measures **attracting measures**

"Attracting measures"

When A is a Lebesgue null set:

- mixing—purely in and of itself—may not be of much physical relevance;
- "attracting" is probably the more physically accessible notion of mixing dynamics [Baladi *et al.*, 2002];
- and yet it seems that attracting measures are under-appreciated and/or under-studied!

E.g. the Lorenz system

- has a chaotic attractor A supporting a physical measure μ_A [Tucker, 2002];
- μ_A is mixing at an exponential rate [Araújo & Melbourne, 2016];
- but is it known whether μ_A is attracting??

What I will do in this talk

I will

- present definitions of attractors, physical measures and attracting measures in a generalised setting;
- present a generalisation of [BR75, Thm. 5.3];
- raise several questions along the way and at the end.

My generalised setting

M – Riemannian manifold

m – Lebesgue measure

d – geodesic distance

$(f^t)_{t \geq 0}$ – continuous semiflow of C^1 local diffeomorphisms

Actually, I can make it even more general (purely topological):

M – Polish space

m – locally finite measure of full support

d – metrisation of the topology of M

$(f^t)_{t \geq 0}$ – continuous semiflow of open mappings $f^t: M \rightarrow M$
admitting a well-defined transfer operator w.r.t. m that locally
respects boundedness.

Physical and attracting measures

A p.m. μ whose support A is compact is called a(n)

- **physical measure** if \exists nbhd $U \supset A$ s.t. for m -a.e. $x_0 \in U$,

$$\frac{1}{t} \int_0^t \delta_{f^s x_0} ds \xrightarrow{\text{weakly}} \mu;$$

- **attracting measure** if \exists nbhd $U \supset A$ s.t. for each p.m. $\nu_0 \ll m$ with $\nu_0(U) = 1$,

$$f^t \nu_0 \xrightarrow{\text{weakly}} \mu.$$

I will generalise the proof of [BR75, Thm. 5.3] to obtain general conditions under which mixing implies attracting.

Attractors

A compact set $A \subset M$ with $f^t A = A$ for all $t \geq 0$ is called a

- **pointwise attractor** if \exists nbhd U s.t.

$$d(f^t x_0, A) \rightarrow 0 \text{ for each } x_0 \in U;$$

- **uniform attractor** if \exists nbhd U s.t.

$$d(f^t x_0, A) \rightarrow 0 \text{ uniformly across } x_0 \in U;$$

- **pointwise attractor via stable manifolds** if \exists nbhd U and $\pi: U \rightarrow A$ [w.l.o.g. **Lebesgue-measurable**] s.t.

$$d(f^t x_0, f^t \pi(x_0)) \rightarrow 0 \text{ for each } x_0 \in U;$$

- **uniform attractor via shadowing** if \exists nbhd U s.t. $\forall \varepsilon > 0$, $\exists \pi_\varepsilon: U \rightarrow A$ s.t. for suff. large t ,

$$\sup_{x_0 \in U} d(f^t x_0, f^t \pi_\varepsilon(x_0)) < \varepsilon.$$

Axiom A case

An Axiom A attractor is both a pointwise via stable manifolds and uniformly via shadowing [BR75, Prop. 4.4].

Questions:

- 1 What about “uniform via stable manifolds”?
 - i.e. does there exist nbhd U and $\pi: U \rightarrow A$ s.t.

$$d(f^t x_0, f^t \pi(x_0)) \rightarrow 0 \text{ uniformly across } x_0 \in U?$$

- 2 Can π (in def'n of pointwise via stable manifolds) be chosen s.t. $\pi(m|_U) \ll \mu_A$?

A trivial result

Proposition

Suppose we have p.m. μ whose support A is a pointwise attractor via stable manifolds.

Suppose (U, π) be chosen s.t. $\pi(m|_U) \ll \mu$.

If μ is mixing then μ is attracting.

Using our further-above characterisation of mixing, the proof is a trivial application of the dominated convergence theorem.

But I suspect that this result is useless (i.e. conditions typically don't hold or are very difficult to verify)??

A maybe more useful result

Theorem (generalising [BR75, Thm. 5.3])

Suppose we have p.m. μ whose support A is a uniform attractor via shadowing.

Suppose μ is mixing. Suppose μ also satisfies $()$:*

\exists arbit'ly small $\varepsilon > 0$ s.t. one can find an unbounded set $\mathcal{T} \subset [0, \infty)$ with

$$\inf_{T \in \mathcal{T}, x \in A} \frac{\mu(y \in A : d(f^t x, f^t y) < \varepsilon \quad \forall t \in [0, T])}{m(y \in M : d(f^t x, f^t y) < 3\varepsilon \quad \forall t \in [0, T])} > 0.$$

Then μ is attracting.

[BR75] uses 2ε in place of 3ε (and verifies $(*)$ in Cor. 4.6), but the proof doesn't seem to work with 2ε .

Some further questions

- Can we find settings outside of the Axiom A setting in which the Theorem can be applied?
- Can the conditions of the Theorem be weakened/modified so as to be more easily applicable beyond the Axiom A setting?
- In particular, might it be the case generally that every mixing physical measure is attracting?

Some further questions

- It is known that if a probability measure μ is an SRB measure in the sense of [Young, 2002], then it is “physical” under a weaker definition where the neighbourhood U may now be any m -positive-measure set [Pugh & Shub, 1989].
- ↪ If a probability measure μ is an SRB measure in the sense of [Young, 2002] and is also mixing, does it follow that μ is an “attracting measure”, at least under a weaker definition where the neighbourhood U may now be any m -positive-measure set?

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Thank you.