Stability index for chaotically driven concave maps

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Overview

- Skew-product systems
- Example: baker's map
- Invariant graph
- Inverse of baker's map and invariant graph
- Stability index

Skew product system

- Consists of base map and fibre map
- We consider the product space $\Theta \times I$

$$F : \Theta \times I \to \Theta \times I,$$

and the skew-product system is given by

$$F(\theta, x) = (S(\theta), \hat{g}(\theta)h(x)). \tag{1}$$

- The first component of equation (1) is independent of x.
- Base map: $S\theta$
- Fibre map: $\hat{g}(\theta)h(x)$

Example: baker's map (base map)

Let $\Theta = [0,1)^2$ and $S : \Theta \to \Theta$ be a baker's transformation

$$S(u,v) = \begin{cases} \left(s^{-1}u, sv\right) & \text{if} \quad u < s \\ \left((1-s)^{-1}(u-s), s + (1-s)v\right) & \text{if} \quad u \geq s \end{cases}$$

NB $S(u,v)$ is invertible.

$\hat{g}(\theta)h(x)$ (fibre map)

- The fibre maps from an interval $I := [0, a]$ into itself of the form:

$$x \mapsto \hat{g}(\theta)h(x)$$

  where $h(x) = arctan(x)$.
- $h(x)$ is invertible, strictly positive, concave and monotonic decreasing.

Invariant graph ($\hat{\varphi}_\infty$)

- General meaning by Keller: is the graph of a *measurable function* from the base space to the fibre space which is *invariant* as a subset of the product space under the skew-product dynamics.
- There is a function $\hat{\varphi}_\infty : \Theta \to I$ which is the invariant graph.
- The global attractor

$$K = \{(\theta, x) : 0 \leq x \leq \hat{\varphi}_\infty(\theta)\}$$

- $\hat{\varphi}_\infty$ is upper semicontinuous.
- 0 is lower semicontinuous.

Invariant graph contd.

- For a skew product system, it is well known that there exists an invariant graph from base space to fibre space whenever the contraction is uniform.
- Stark (1997) has studied the invariant graphs for forced system while Glendinning (2002) studied such the graphs in a pinched skew product.
- Stark proved the existence of a continuous invariant graph for the skew product.
- Glendinning proved that the global attractor K lies between an upper semicontinuous curve and a lower semicontinuous curve.

Mapping of invariant graph with baker's map

$$F\begin{pmatrix} u \\ v \\ x \end{pmatrix} \rightarrow \begin{pmatrix} S \\ g(v)h(x) \end{pmatrix}, \tag{2}$$

with
$S = S_1(u, v) = (\frac{u}{s}, sv)$ if $0 \leq u < s$,
$S = S_2(u, v) = (\frac{u-s}{1-s}, s + (1-s)v)$ if $s \leq u < 1$,
$g(v) = r(1 + \varepsilon + \cos(2\pi v))$,
$h(x) = \arctan(x)$,
where $s = 0.45, \varepsilon = 0.01, r = 2.5$.

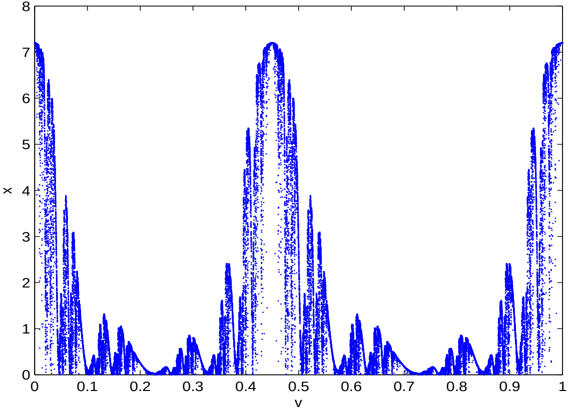


Figure: The attractor which is the invariant graph $\hat{\varphi}_\infty(v)$ for the baker's map x vs. v for $r = 2.5$.

Attractor (3d)

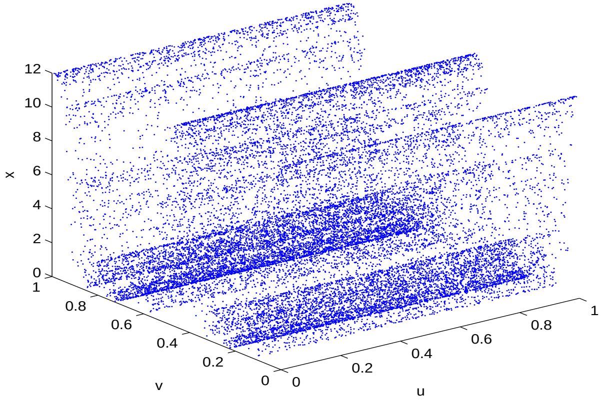


Figure: The three-dimensional invariant graph for the baker's map u, v, x.

Inverse of baker's map and invariant graph

Let $G = F^{-1}$ and by ignoring the $(')$ sign, the inverse maps are:
i) If $0 \leq v < s$, then

$$G(u, v, x) = \left(su, s^{-1}v, \tan\left(\frac{x}{g(\frac{v}{s})} \right) \right).$$

ii) If $s \leq v < 1$, then

$$G(u, v, x) = \left((1-s)u + s, (1-s)^{-1}(v-s), \tan\left(\frac{x}{g(\frac{v-s}{1-s})} \right) \right).$$

By using this inverse map, we will plot the basin of attraction in the journey to compute the stability index.

Basin of attraction for Keller's map

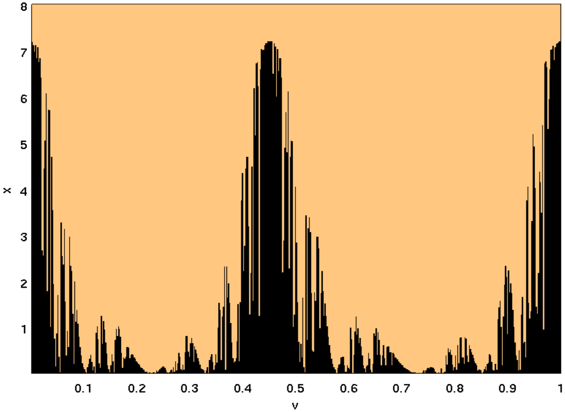


Figure: The basin of attraction for the inverse of invariant graph and the baker's map for $r = 2.5$. The black area denotes the basin where the points go to $x = 0$ and the orange area denotes the points go to $x = \infty$.

Comparison

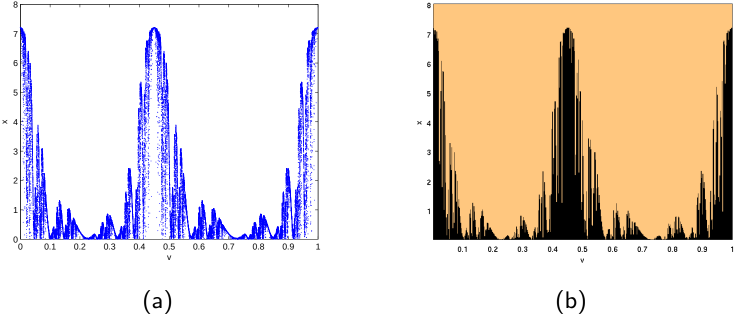


Figure: $r = 2.5$. (a) The attractor invariant graph $\hat{\varphi}_\infty(v)$ for baker's map. (b) The basin of attraction for the inverse of invariant graph and the baker's map. As we iterate backward, the point will either goes to 0 or ∞.

What is stability index?

- Introduced by Podvigina and Ashwin (2011).
- Consider a point $x \in X$, and defined that
$$\Sigma_\epsilon(x) = \frac{\ell(B_\epsilon(x)\cap N)}{\ell(B_\epsilon(x))},$$
where $B_\epsilon(x)$ is a ball of radius ϵ about x, N is basin of attraction and $\ell(\cdot)$ denotes Lebesgue measure on $\mathbb{R}^n$.
- Note that $0 \leq \Sigma_\epsilon(x) \leq 1$ and define that $\sigma(x) = [-\infty, \infty]$.

Definition

For a point $x \in X$, the stability index of X at x to be
$$\sigma(x) := \sigma_+(x) - \sigma_-(x),$$
which exists when the following converge:
$$\sigma_-(x) := \lim_{\epsilon \to 0} \frac{\ln(\Sigma_\epsilon(x))}{\ln \epsilon}, \quad \sigma_+(x) := \lim_{\epsilon \to 0} \frac{\ln(1 - \Sigma_\epsilon(x))}{\ln \epsilon}.$$

- $\sigma(x)$ of a point $x \in X$ characterizes the local geometry of the basin of attraction of X.

Stability index $\sigma(v)$

- Keller computed $\sigma(v)$ for the global attractor K for F.
- He pick a point $(v, 0)$ on invariant graph and define a local stability index $\sigma(v)$ in the following way:

$$\sigma(v) = \sigma_+(v) - \sigma_-(v),$$

where

$$\sigma_-(v) = \lim_{\epsilon \to 0} \frac{\log \Sigma_\epsilon(v)}{\log \epsilon}, \quad \sigma_+(v) = \lim_{\epsilon \to 0} \frac{\log(1 - \Sigma_\epsilon(v))}{\log \epsilon},$$

with

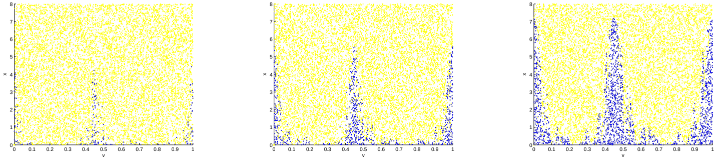
$$\Sigma_\epsilon(v) = \frac{1}{\epsilon |U_\epsilon(v)|} \int_{U_\epsilon(v)} \min\{\varphi_\infty(t), \epsilon\} dt$$

and

$$1 - \Sigma_\epsilon(v) = \frac{1}{\epsilon |U_\epsilon(v)|} \int_{U_\epsilon(v)} (\epsilon - \varphi_\infty(t))^+ dt.$$

$U_\epsilon(v) = (v - \epsilon, v + \epsilon)$ with size 2ϵ-symmetric interval nbhd.
- In our case, we aim to compute $\sigma(v)$ for the basin of attraction 0 for F^{-1}.

- This is computing using F^{-1}.



(a) $r = 1.74$ (b) $r = 2.0$ (c) $r = 2.5$

Figure: The basin of attraction by using random number generator. The blue dots denote the basin where the points go to $x = 0$ and the yellow dots denote the points go to $x = \infty$.

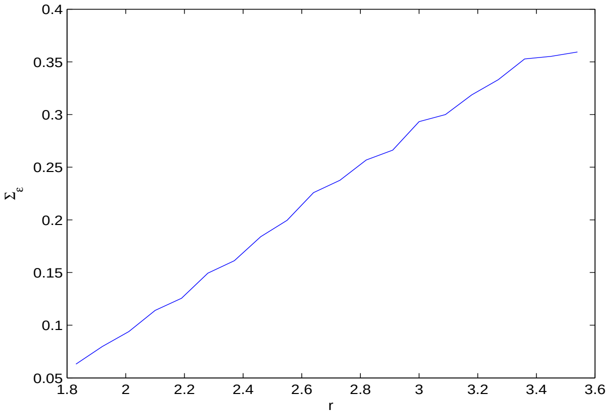


Figure: The proportion of the blue dots over the whole image for $r = 1, \ldots, 5$. We can see that the proportion is increase as we increase the parameter r.

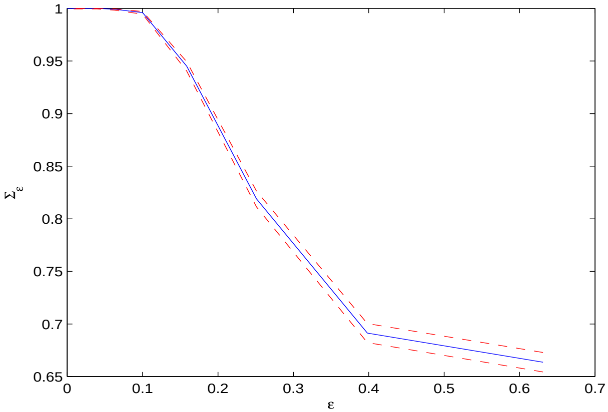


Figure: The proportion of the blue dots over the ϵ for $r = 2.5$. The proportion decrease with the increase of ϵ. The proportion 1 means that the neighbourhood only contains the blue points whereas the proportion < 1 shows that the blue and yellow points are mixed together.

$\sigma_+(v)$

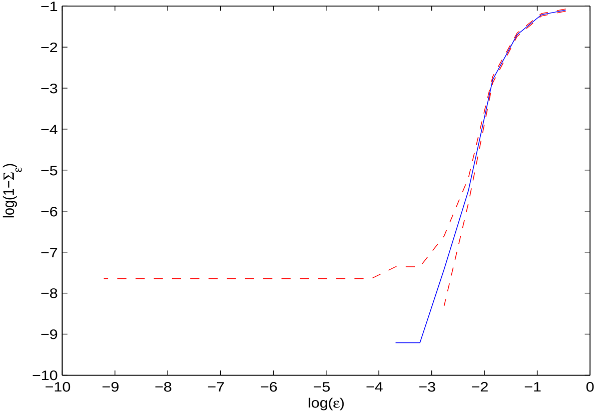


Figure: Computation of $\sigma_+(v)$: $\log(1 - \Sigma_\epsilon(v))$ vs. $\log(\epsilon)$ for $r = 2.5$. Here the slope is ∞.

$\sigma_-(v)$

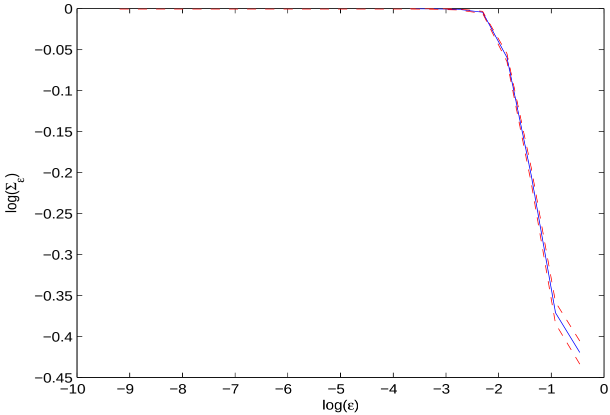


Figure: Computation of $\sigma_-(v)$: $\log(\Sigma_\epsilon(v))$ vs. $\log(\epsilon)$ for $r = 2.5$. Here the slope is 0.

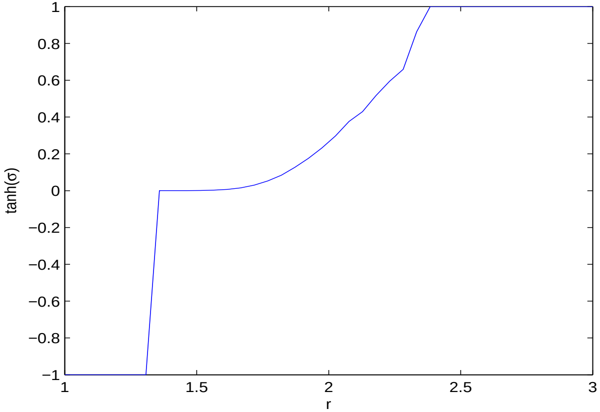


Figure: Stability index for Keller's paper for $r = 1, \dots, 3$. Here the index ranges from $-\infty$ to ∞ where in this figure -1 represents the $-\infty$ and 1 represents ∞. Therefore $\sigma(v) = [-\infty, \infty]$.

Future work

- Generalize stability index rather than just for a point.
- Use stability index to characterize the invariant graphs in terms of Lyapunov exponents in the case of riddled basins.

References

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THANKS FOR LISTENING!