Non-Autonomous Instabilities: Interactions Between Noise and Rate-Induced Tipping

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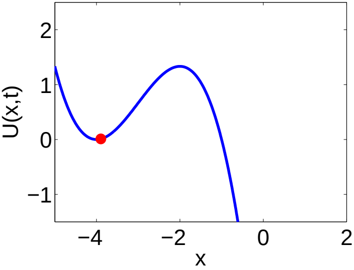
Outline

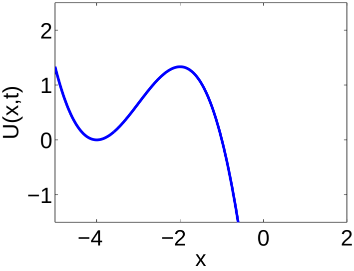
Prototype model for rate-induced tipping (S. Wieczorek)

- A Tipping event occurs when gradual changes to input levels causes the system to change states.

$$\dot{x} = f(x,t) = (x+\lambda(t))^2 - 1, \qquad U(x,t) = -\int_x f(\bar{x},t)\mathrm{d}\bar{x}$$



Rate-induced tipping not escaping well



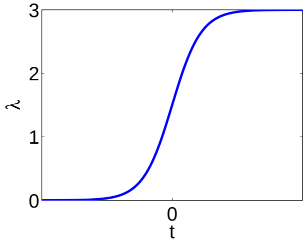
Rate-induced tipping escaping well

Normal form for rate-induced tipping

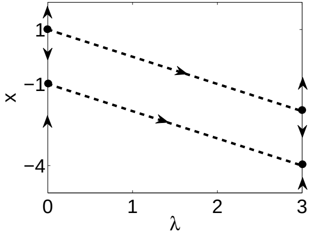
- Simplest model for rate-induced tipping:

$$\dot{x} = (x+\lambda)^2 - 1$$
$$\dot{\lambda} = \epsilon\lambda(\lambda_{max} - \lambda)$$
$$\lambda(t) = \frac{\lambda_{max}}{2}\left(\tanh\left(\frac{\lambda_{max}\epsilon t}{2}\right) + 1\right)$$

(Ashwin et al., 2012)



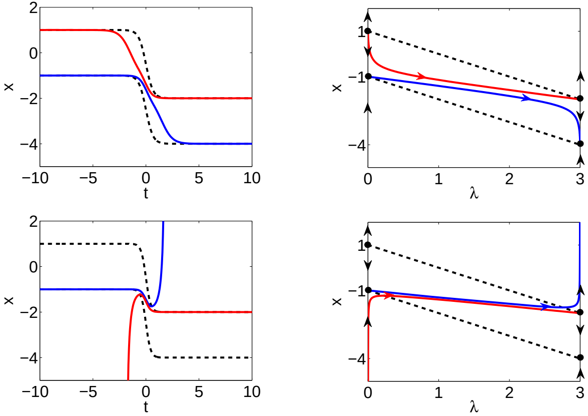
Ramping parameter λ



Phase plane, $\epsilon \approx 0$

Prototype model for rate-induced tipping (S. Wieczorek)

$$\dot{x} = (x+\lambda)^2 - 1, \qquad \dot{\lambda} = \epsilon\lambda(\lambda_{max} - \lambda)$$



$\epsilon < \epsilon_c$

$\epsilon > \epsilon_c$

Time profile

Phase plane

Adding Noise

- Brownian motion of a particle is governed by the stochastic differential equation:

$$\mathrm{d}X_t = f(X_t, t)\mathrm{d}t + \sqrt{2D}\mathrm{d}W_t$$

drift $f(X_n, t_n)$; D diffusion.

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- This is can be discretised into the form:

$$X_{n+1} = X_n + f(X_n, t_n)\mathrm{d}t + \eta\sqrt{2D}\sqrt{\mathrm{d}t}$$

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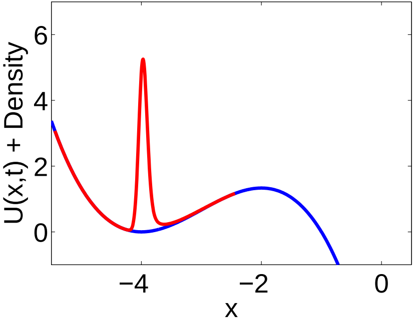
- For the rate-induced example this becomes:

$$X_{n+1} = X_n + ((X_n + \lambda(t_n))^2 - 1)\mathrm{d}t + \eta\sqrt{2D}\sqrt{\mathrm{d}t}$$

Fokker-Planck Equation (FPE)

- Probability density function of the random variable X_t is governed by the Fokker-Planck equation (FPE):

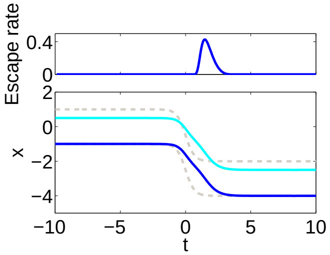
$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2} - \frac{\partial}{\partial x}(f(x,t)P(x,t))$$



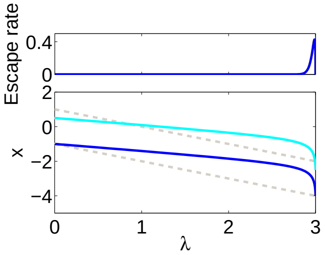
How density $P(x,t)$ evolves in potential well $U(x,t)$

Noise and rate-induced tipping

Time profile and phase plane of rate-induced tipping along with the escape rate, $\epsilon = 1.25$, $D = 0.008$, Prob. of escape $= 0.45$



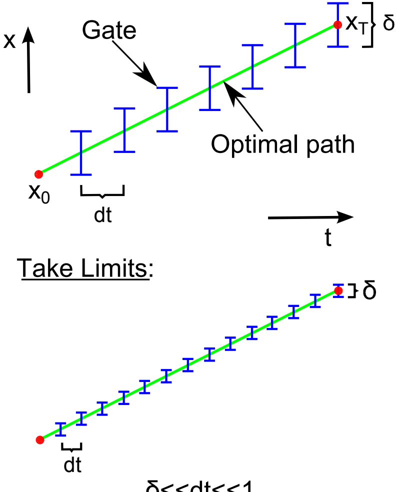
Time profile



Phase plane

Optimal path definition

The optimal path is the most likely path for getting from x_0 to x_T in a time T whilst remaining within the gates of the path.



Variational problem for the optimal path

From the FPE it turns out that we need to maximise the following functional F:

$$F = \exp\left[\frac{U(x_0, t_0) - U(x_T, T)}{2D} - \int_{t_0}^{T} \left(\frac{\dot{x}^2}{4D} + V_s\right) \mathrm{d}\tau\right]$$

which gives us a Boundary Value Problem (BVP):

$$\ddot{x} = 2D\frac{\partial V_s}{\partial x}, \qquad \begin{cases} x(0) & = x_0 \\ x(T) & = x_T \end{cases}$$

where

$$V_s = \frac{1}{4D}\left(\frac{\partial U}{\partial x}\right)^2 - \frac{1}{2}\frac{\partial^2 U}{\partial x^2} - \frac{1}{2D}\frac{\partial U}{\partial t}$$

(Zhang, 2008), (Ho and Dai, 2008)

Variational problem for the optimal path

BVP:

$$\ddot{x} = 2D\frac{\partial V_s}{\partial x}, \qquad \begin{cases} x(0) &= x_0 \\ x(T) &= x_T \end{cases}$$

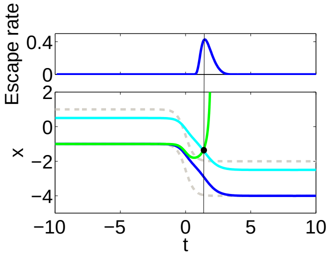
- To find the optimal time for our optimal path we maximise our F again by keeping T_{end} free:

$$F = \exp\left[\frac{U(x_0, t_0) - U(x_T, T_{end})}{2D} - \int_{t_0}^{T_{end}} \left(\frac{\dot{x}^2}{4D} + V_s\right) \mathrm{d}\tau\right]$$

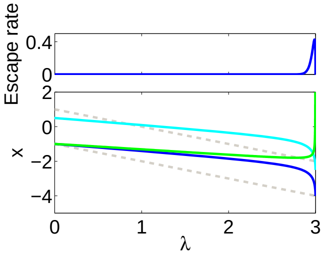
this is performed using continuation techniques in AUTO.

Optimal path for rate-induced tipping

Optimal path of escape for rate-induced tipping along with the escape rate, $\epsilon = 1.25$, $D = 0.008$, Prob. of escape $= 0.45$



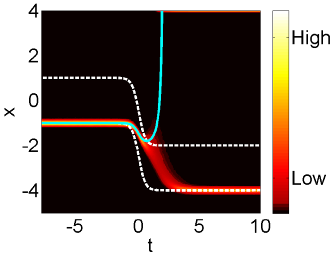
Time profile



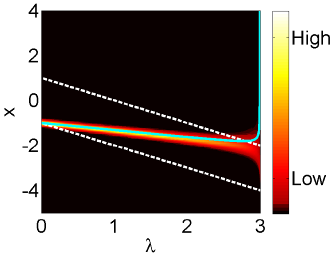
Phase plane

Density plots from simulations

Density plots of simulations started at $x_0 = -1$ at $t = -10$ and run until $t = 10$ for rate-induced system with optimal path added, $\epsilon = 1.25$, $D = 0.008$



Time profile

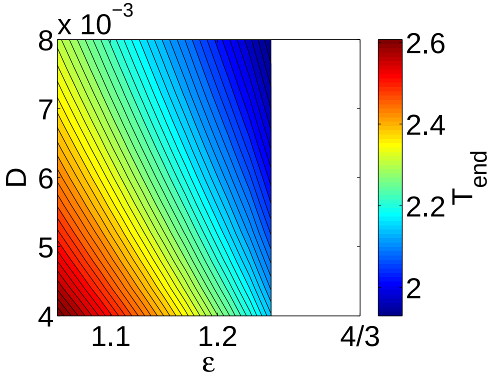


Phase plane

Colour plot for optimal time

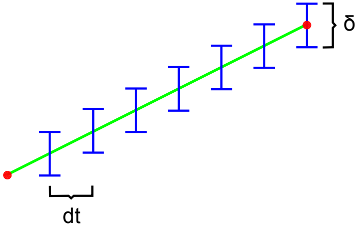
- Maximising the functional F using continuation techniques in AUTO gives the optimal time for escape.

Colour contour plots for the optimal time of escape for a range of ϵ and D values.



Calculating probability of following a path

Previously,



This time we take $dt \to 0$ but keep δ fixed.

- Calculate probability of going from one gate to the next assuming we are in the gate to start with.
- Use instantaneous eigenmodes of the system to approximate the probability of escape.

Instantaneous Eigenmodes for FPE

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- Fokker-Planck equation:

$$\frac{\partial P(x,t)}{\partial t} = \left[D\frac{\partial^2}{\partial x^2} - f(x,t)\frac{\partial}{\partial x} - \frac{\partial f(x,t)}{\partial x} \right] P(x,t)$$

- The FPE can then be written as:

$$\dot{\mathbf{P}} = A(t)\mathbf{P}$$

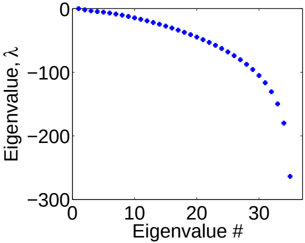
- Assume solution to be of the form:

$$\mathbf{P}(t) = x_1(t)\mathbf{v}_1(t) + x_2(t)\mathbf{v}_2(t) + \ldots$$

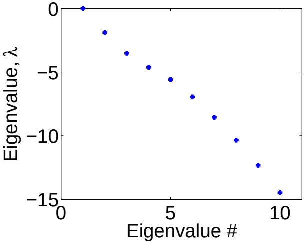
where $A(t)\mathbf{v}_k(t) = \lambda_k(t)\mathbf{v}_k(t)$

Instantaneous Eigenvalue Spectrum

$$A(t) = D\frac{\partial^2}{\partial x^2} - f(x,t)\frac{\partial}{\partial x} - \frac{\partial f(x,t)}{\partial x}$$



Full spectrum



10 dominant eigenvalues

Eigenvalue spectrum for the full rate-induced system

Instantaneous Eigenmodes for FPE

- Initial x_k given by projection of some given initial density:

$$x_k = < \mathbf{w}_k^T, \mathbf{P}^{initial} >$$

where $\mathbf{w}_k^T$ arise from the adjoint of the matrix A:

$$A^{adj}(t) = D\frac{\partial^2}{\partial x^2} + f(x,t)\frac{\partial}{\partial x}$$

- Subsequent x_k are gained through substituting the assumed solution into the FPE:

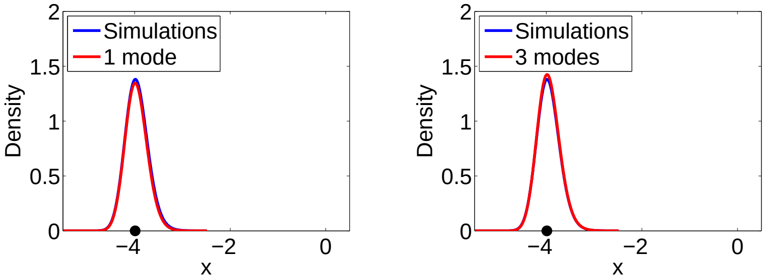
$$\begin{aligned} \dot{x}_1\mathbf{v}_1 + x_1\dot{\mathbf{v}}_1 + \dot{x}_2\mathbf{v}_2 + x_2\dot{\mathbf{v}}_2 + \ldots &= A(x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots) \\ &= \lambda_1 x_1\mathbf{v}_1 + \lambda_2 x_2\mathbf{v}_2 + \ldots \end{aligned}$$

- Multiply this equation on the left by $\mathbf{w}_1^T$ and use $\mathbf{w}_i^T\mathbf{v}_j = \delta_{ij}$ to give:

$$\dot{x}_1 = \lambda_1 x_1 - \mathbf{w}_1^T\dot{\mathbf{v}}_1 x_1 - \mathbf{w}_1^T\dot{\mathbf{v}}_2 x_2 - \ldots$$

Comparison of simulations with eigenmodes

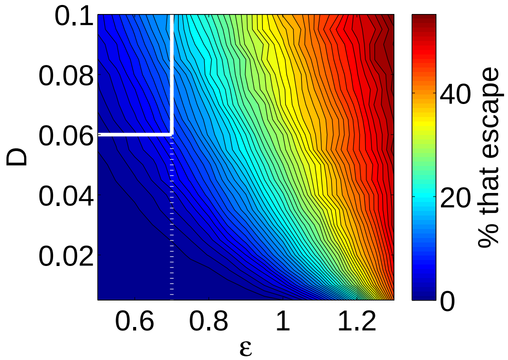
Parameter values: $\epsilon = 0.7$, $D = 0.1$



Comparison between using simulations and 1 eigenmode

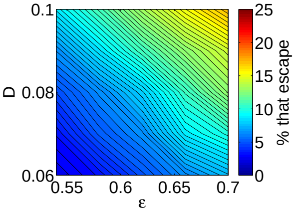
Comparison between using simulations and 3 eigenmodes

Overview of probability of escape using simulations

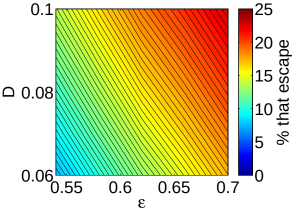


Starting simulations at $x_0 = -1$ and observing the probability of escaping potential well for a large range of ϵ and D values

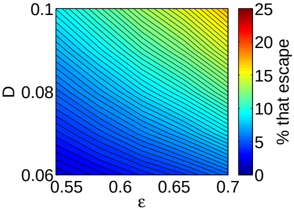
Comparison for probability of escape using different techniques



Simulations



1 mode



3 modes

Contour plots comparing % that escape denoted by the colour in using simulations with using either 1 or 3 modes.

Summary

- To calculate the timing of escape we have a BVP that can be solved.
- For probability of escape we can use the mode approximation of the system.
- The timing and probability of escape can be used as an extra early-warning indicator for tipping events.

References

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