Dynamics Reading Group
Most likely paths

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Introduction

- Consider a deterministic ODE:

$$\dot{x} = \mu(x(t))$$

  For a given starting position, end point and time there is one solution.

- But if we were to add noise to this then suddenly there are many different paths possible and we're interested in finding the most likely path.

Introduction

- Objectives:
  - Calculate the most likely path from a known starting position to a known end position in a given time.
  - What is the probability of following this path?
  - Is there an optimum time to make the transition?
- Brownian motion of a particle is governed by the stochastic differential equation:

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t + \sqrt{2D(X_t, t)}\mathrm{d}W_t$$

with drift $\mu(X_t, t)$ and $D(X_t, t)$ is a diffusion coefficient.

Introduction

- Probability density function of the random variable X_t is governed by the Fokker-Planck equation:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}(\mu(x,t)P(x,t)) + \frac{\partial^2}{\partial x^2}(D(x,t)P(x,t))$$

where a potential well $U(x,t)$ satisfies:

$$\frac{\mathrm{d}U(x,t)}{\mathrm{d}x} = -\mu(x,t)$$

Introduction

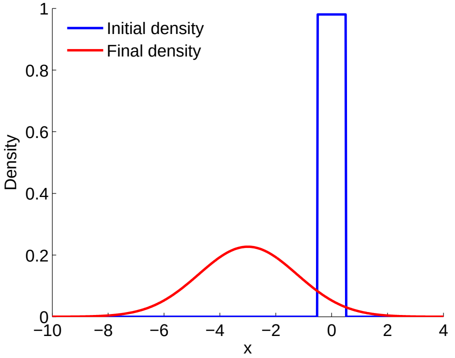

Figure : Fokker-Planck run for $\dot{x} = -1 + \eta$, $x_0 = 0$, $T = 3$

Key book results

- Case 1: Pure diffusion

$$\tilde{P} \propto \exp\left(-\frac{1}{4D}\int_{t_0}^{T}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 \mathrm{d}\tau\right)$$

Key book results

- Case 1: Pure diffusion

$$\tilde{P} \propto \exp\left(-\frac{1}{4D} \int_{t_0}^{T} \left(\frac{\mathrm{d}x}{\mathrm{d}\tau} \right)^2 \mathrm{d}\tau \right)$$

- Case 2: annihilation model

$$\tilde{P} \propto \exp\left(-\frac{1}{4D} \int_{t_0}^{T} \left(\frac{\mathrm{d}x}{\mathrm{d}\tau} \right)^2 \mathrm{d}\tau - \int_{t_0}^{T} A(x(\tau), \tau)\mathrm{d}\tau \right)$$

Key book results

- Case 3: Most likely path will maximise:

$$W(x(\tau)) = \exp\left(\frac{U(x_0) - U(x_T)}{2D}\right)\exp\left(-\int_{t_0}^{T}\mathbb{L}(x(\tau))\mathrm{d}\tau\right)$$

where

$$\mathbb{L} = \frac{1}{4D}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + V_s$$

and

$$V_s = \frac{1}{4D}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2 - \frac{1}{2}\frac{\mathrm{d}^2U}{\mathrm{d}x^2}$$

Key book results

In order to minimise need to solve the Euler-Lagrange equation:

$$\frac{\partial \mathbb{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \mathbb{L}}{\partial \dot{x}} = 0$$

From this a 2^{nd} order ODE is created that the most likely trajectory will satisfy:

$$\ddot{x} = 2D\frac{\mathrm{d}V_s}{\mathrm{d}x}$$

Example

Consider the example:

$$\mathrm{d}x_t = -ax_t\mathrm{d}t + \sigma\mathrm{d}W_t$$

Most likely path satisfies:

$$\ddot{x} - a^2x = 0$$

In this case the solution can be obtained analytically:

$$x(t) = x_0\mathrm{e}^{at} + (x_T - x_0\mathrm{e}^{aT})\frac{\sinh(at)}{\sinh(aT)}$$

Example

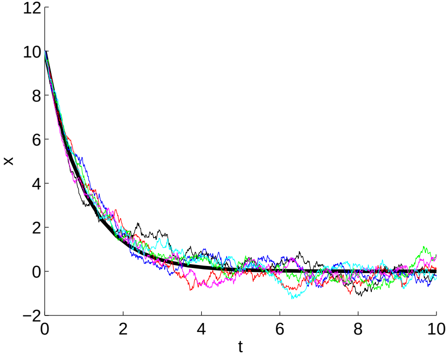

Figure : Most likely path for constant single potential well, $a = 1, x_0 = 10, x_T = 0, T = 10$

Time dependent potential wells

- Time dependent potential wells of the form $U(x,t)$
- New PDE is:

$$\frac{\partial P_s}{\partial t} = D\frac{\partial^2 P_s}{\partial x^2} + \left(\frac{U''}{2} - \frac{U'^2}{4D} + \frac{\dot{U}}{2D}\right)P_s$$

- 2^{nd} order ODE remains the same with just an extra term in V_s:

$$\ddot{x} = 2D\frac{\mathrm{d}V_s}{\mathrm{d}x}$$

Example

- Suppose now a is not constant, $a(t) = a_0 - \epsilon t$, with drift speed ϵ
- The most likely path now satisfies:

$$\ddot{x} = (a(t)^2 + \epsilon)x$$

- To be solved numerically

Example

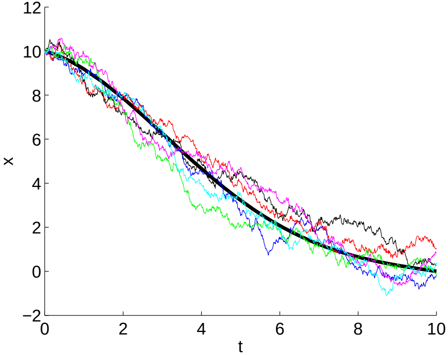

Figure : Most likely path for time dependent single potential well, $a_0 = 0.05, \epsilon = -0.07, x_0 = 10, x_T = 0,\ T = 10$

Example

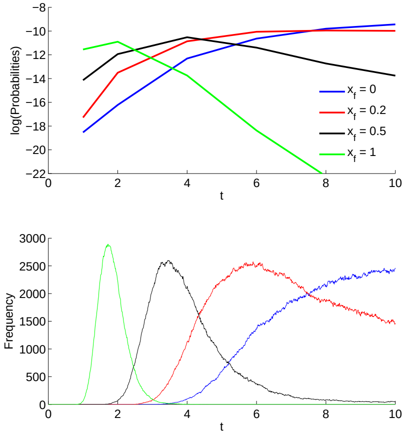

Figure : Optimal time for constant single potential well, $a = 0.4, D = 0.01, x_0 = 2$

References

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