Restrictedly invariant measures

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TIPPING POINTS IN THE EARTH SYSTEM

Abstract

Classical ergodic theory assumes that the pre-image of every measurable set is measurable, and considers measures that are perfectly preserved by the dynamics. I have recently been thinking about the scenario where one only has a limited-resolution view of the state space – formalised as a sub-sigma-algebra of sets whose pre-images do not necessarily belong to the same sub-sigma-algebra. And I have been considering measures that are preserved only on this sub-sigma-algebra. My original motivation for this comes from the aim to find the most natural formalism for stationary measures of coloured-noise-driven processes within the setting of Ludwig Arnold's abstract framework for random dynamical systems. I will present some of my preliminary results on this topic.

Invariant and ergodic measures

$(X, \mathcal{X})$ – measurable space
$f\colon X \to X$ an $(\mathcal{X}, \mathcal{X})$-measurable map

An **invariant measure of** f (or f**-invariant measure**):
probability measure μ on $(X, \mathcal{X})$, s.t.

$$\mu(f^{-1}(A)) \;=\; \mu(A) \quad \forall\, A \in \mathcal{X}.$$

Physical interpretation:

- let ξ be a random variable taking values in X, and
- suppose the probability distribution of ξ is μ;
- then the random variable $f(\xi)$ has the same probability distribution μ.

Invariant and ergodic measures

An **ergodic measure of** f (or f**-ergodic measure**):
f-invariant measure μ for which the following equivalent statements hold,

- any $A \in \mathcal{X}$ with $f^{-1}(A) = A$ has $\mu(A) = 0$ or 1;
- any $A \in \mathcal{X}$ with $f^{-1}(A) \overset{\mathrm{mod}\,\mu}{=} A$ has $\mu(A) = 0$ or 1.

Heuristically: you can't split X into two μ-non-trivial components A and $X \setminus A$ that "stay separate" under f.

Markov operators

Given $f\colon X \to X$ with invariant measure μ, we can define the associated "Koopman operator" $P_{f,\mu}$ (defined shortly).

Koopman operators are the "trivial" or "deterministic" case of **Markov operators.**

MOs are a way to describe **"transition probabilities"** for a random process that takes you **from "the current" state to "the next" state**. . .

Markov operators

Space of possible states:

- measurable space $(X, \mathcal{X})$,
- equipped with a probability measure μ, which represents the probability distribution for your "current" state.

Some notation:

- Let $\mathcal{L}^1(X, \mathcal{X}, \mu) \ = \ \{g \colon X \to \mathbb{R} \ : \ \int_X |g| \, d\mu < \infty\}$.
- For each $g \in \mathcal{L}^1(X, \mathcal{X}, \mu)$, let

$$[g]_\mu = \{\tilde{g} \colon X \to \mathbb{R} \ : \ \tilde{g} \overset{\mu\text{-a.s.}}{=} g\}.$$

- Let $L^1(X, \mathcal{X}, \mu) \ = \ \{[g]_\mu : g \in \mathcal{L}^1(X, \mathcal{X}, \mu)\}$.

We define $\int_X [g]_\mu \, d\mu = \int_X g \, d\mu$. Also, $L^1(X, \mathcal{X}, \mu)$ is a vector space, under the natural operations defined such that

$$\lambda_1 [g_1]_\mu + \lambda_2 [g_2]_\mu \ = \ [\lambda_1 g_1 + \lambda_2 g_2]_\mu.$$

Markov operators

A Markov operator is a function $P\colon L^1(X,\mathcal{X},\mu) \to L^1(X,\mathcal{X},\mu)$ satisfying certain rules described shortly; heuristically:

- for $A \in \mathcal{X}$, for $h \in P([\mathbb{1}_A]_\mu)$, $h(x)$ represents

$$\mathrm{Prob}(\text{ next state} \in A \mid \text{current state} = x\ );$$

- more generally, for $h \in P([g]_\mu)$, $h(x)$ represents

$$\mathbb{E}[\ g(\text{next state}) \mid \text{current state} = x\ ].$$

These statements are to be understood *"for μ-almost all x"*. So Markov operators only give a **"fuzzy description" of transition probabilities.**

Markov operators: the definition

Given a prob. space $(X, \mathcal{X}, \mu)$, a **Markov operator on $L^1(X, \mathcal{X}, \mu)$** is a function $P \colon L^1(X, \mathcal{X}, \mu) \to L^1(X, \mathcal{X}, \mu)$ satisfying the four requirements:

- linearity – $P(\lambda_1 g_1 + \lambda_2 g_2) = (\lambda_1 P g_1) + (\lambda_2 P g_2)$;
- monotonicity – $g \overset{\mu\text{-a.s.}}{\geq} 0 \;\Rightarrow\; Pg \overset{\mu\text{-a.s.}}{\geq} 0$;
- preserves constant functions – $P([1]_\mu) = [1]_\mu$;
- for each $g \in L^1(X, \mathcal{X}, \mu)$,

$$\int_X Pg \, d\mu = \int_X g \, d\mu.$$

This last point means: **prior to the knowledge of the current state, the probability distribution of the next state is the same as the probability distribution of the current state.**

Ergodic Markov operators

A Markov operator P on $L^1(X, \mathcal{X}, \mu)$ is **ergodic** if any $A \in \mathcal{X}$ with

$$P([\mathbb{1}_A]_\mu) = [\mathbb{1}_A]_\mu$$

has $\mu(A) = 0$ or 1.

Heuristic meaning of "$P([\mathbb{1}_A]_\mu) = [\mathbb{1}_A]_\mu$":

$$\begin{aligned} \mathrm{Prob}(\text{ next state} \in A \mid \text{current state} \in A\ ) &= 1 \\ \mathrm{Prob}(\text{ next state} \in A \mid \text{current state} \notin A\ ) &= 0. \end{aligned}$$

Koopman operators

Heuristically, a Koopman operator is a MO for which **the process taking you from “the current” state to “the next” state is a deterministic map.**

Koopman operators

measurable space $(X, \mathcal{X})$
measurable map $f \colon X \to X$
f-invariant measure μ

The corresponding **Koopman operator** is the Markov operator $P_{f,\mu}$ on $L^1(X, \mathcal{X}, \mu)$ defined by

$$P_{f,\mu}([g]_\mu) \;=\; [g \circ f]_\mu \quad \forall\, g \in \mathcal{L}^1(X, \mathcal{X}, \mu).$$

(I will now start to drop notational distinctions between functions and their μ-equivalence classes.) So for $A \in \mathcal{X}$,

$$(P_{f,\mu}\mathbb{1}_A)(x) \;=\; \text{“Prob(next state} \in A \mid \text{current state} = x \text{)”}$$

$$\overset{\mu\text{-a.s.}}{=} \begin{cases} 1 & f(x) \in A \\ 0 & f(x) \notin A. \end{cases}$$

Invariance and ergodicity in terms of KOs

Remark. Given $(X, \mathcal{X})$, measurable map $f\colon X \to X$, and an arbitrary probability measure μ,

- we can define a Markov operator $P_{f,\mu}$ on $L^1(X, \mathcal{X}, \mu)$ by

$$P_{f,\mu}([g]_\mu) \;=\; [g \circ f]_\mu \quad \forall\, g \in \mathcal{L}^1(X, \mathcal{X}, \mu)$$

  if and only if μ is f-invariant;

- assuming μ is f-invariant, the Koopman operator $P_{f,\mu}$ is ergodic **if and only if** μ is f-ergodic.

Sub-σ-algebras: "limited access to detail"

Given $(X, \mathcal{X})$, a **sub-σ-algebra** of $\mathcal{X}$ is a σ-algebra $\tilde{\mathcal{X}}$ on X with $\tilde{\mathcal{X}} \subset \mathcal{X}$.

$\rightarrow$ $\tilde{\mathcal{X}}$ represents a "lower-resolution description" of the set of states X.

Example scenarios:

- $(X, \mathcal{X})$ is the space of all possible states of the global climate; but $\tilde{\mathcal{X}}$ represents just a finite number of variables describing the climate, e.g. global mean temperature and global mean CO_2 concentration.
- $(X, \mathcal{X})$ is the space of all possible bi-infinite-time paths of a random walk on a compact group; but $\tilde{\mathcal{X}}$ represents just the past and present, not the unseen future.

"Non-trivial" transition probabilities

Given map $f\colon X \to X$ and prob. meas. μ on $(X, \mathcal{X})$,

- $\rightarrow$ suppose "we can only see" the sub-σ-algebra $\tilde{\mathcal{X}}$;
- $\rightarrow$ then the "transition probabilities"

$$\mu(f^{-1}(A) \,|\, \tilde{\mathcal{X}})(x), \quad x \in X, A \in \tilde{\mathcal{X}}$$

  are no longer "deterministic", i.e. they can take values in the open interval $(0, 1)$.

E.g. if I can only see that this year's average GMT is 14 °C, the probability that next year's average GMT will lie between 14 °C and 14.5 °C is not 0 or 1 – even if the underlying dynamics of the global climate is deterministic!

"Non-trivial" transition probabilities

Question: *When can we describe these "transition probabilities" by a Markov operator?*

I.e. when is there a Markov operator $P_{f,\mu,\tilde{\mathcal{X}}}$ on $L^1(X, \tilde{\mathcal{X}}, \mu|_{\tilde{\mathcal{X}}})$ such that

$$P_{f,\mu,\tilde{\mathcal{X}}} g \;=\; \mathbb{E}_\mu[g \circ f | \tilde{\mathcal{X}}]$$

for all $\tilde{\mathcal{X}}$-measurable functions $g \colon X \to \mathbb{R}$ with $\int_X |g| \, d\mu < \infty$?

Answer. This Markov operator exists **if and only if**

$$\mu(f^{-1}(A)) \;=\; \mu(A) \quad \forall\, A \in \tilde{\mathcal{X}}.$$

Restrictedly invariant and ergodic measures

Let

- $(X, \mathcal{X})$ be a measurable space,
- $\tilde{\mathcal{X}}$ be a sub-σ-algebra of $\mathcal{X}$,
- $f\colon X \to X$ be an $(\mathcal{X}, \tilde{\mathcal{X}})$-measurable function.
  - $\rightarrow$ In practice, f will be $(\mathcal{X}, \mathcal{X})$-measurable, but this is stronger than needed.

An $\tilde{\mathcal{X}}$**-restrictedly** f**-invariant measure** is a probability measure μ on $(X, \mathcal{X})$ s.t.

$$\mu(f^{-1}(A)) \;=\; \mu(A) \quad \forall\, A \in \tilde{\mathcal{X}}.$$

Physical interpretation:

- let ξ be a random variable taking values in X, and
- suppose the probability distribution of ξ is μ;
- then $\mathrm{Prob}(\, f(\xi) \in A\,) = \mathrm{Prob}(\, \xi \in A\,) = \mu(A)$ for all $A \in \tilde{\mathcal{X}}$.

Restrictedly invariant and ergodic measures

Given $\tilde{\mathcal{X}}$-restrictedly f-invariant measure μ, the following are equivalent:

- the "Koopman" operator $P_{f,\mu,\tilde{\mathcal{X}}}$ on $L^1(X, \tilde{\mathcal{X}}, \mu|_{\tilde{\mathcal{X}}})$,

$$P_{f,\mu,\tilde{\mathcal{X}}}g = \mathbb{E}_\mu[g \circ f | \tilde{\mathcal{X}}],$$

  is ergodic;
- any $A \in \tilde{\mathcal{X}}$ with $f^{-1}(A) \overset{\mathrm{mod}\,\mu}{=} A$ has $\mu(A) = 0$ or 1.

An $\tilde{\mathcal{X}}$**-restrictedly** f**-ergodic measure** is an $\tilde{\mathcal{X}}$-restrictedly f-invariant measure μ for which the above equivalent statements hold.

Remark. Suppose f is $(\mathcal{X}, \mathcal{X})$-measurable. Then every f-invariant measure is $\tilde{\mathcal{X}}$-restrictedly f-invariant, and every f-ergodic measure is $\tilde{\mathcal{X}}$-restrictedly f-ergodic.

I asked on MathOverflow (`https://mathoverflow.net/questions/351548`) about whether these "restrictedly invariant measures" have ever been studied before. This was over 2 months ago, and I have still received no answers (but also no upvotes).

There are potentially many questions that one can ask about RIMs; one that I have considered is the "structure" of the set of $\tilde{\mathcal{X}}$-RIMs of a map f.

Convex sets and extreme points

Let V be a real vector space.

- An **open line segment** is a set of the form

$$L_{\{\mathbf{u},\mathbf{v}\}} := \{\lambda\mathbf{u} + (1-\lambda)\mathbf{v} \,:\, \lambda \in (0,1)\}$$

  for two *distinct* points $\mathbf{u}, \mathbf{v} \in V$.

- A set $C \subset V$ is **convex** if for any two distinct points $\mathbf{u}, \mathbf{v} \in C$, we have $L_{\{\mathbf{u},\mathbf{v}\}} \subset C$.
  $\rightarrow$ *Important:* Vacuously, a singleton is convex!

- An **extreme point** of a convex set $C \subset V$ is a point $\mathbf{x} \in C$ such that there does not exist an open line segment L with $\mathbf{x} \in L \subset C$.

Write $\mathrm{extr}(C) := \{\text{extreme points of } C\}$.

Convex sets and extreme points

Examples:
(a) closed solid square; (b) closed disk; (c) open disk.

Extreme points: (a) the four corners; (b) all points on the circumference; (c) no extreme points.

Affine maps

Let V, W be real vector spaces;
let $C_V \subset V$ and $C_W \subset W$ be convex sets.

A function $T\colon C_V \to C_W$ is **affine** if

$$T(\lambda \mathbf{u} + (1-\lambda)\mathbf{v}) = \lambda T(\mathbf{u}) + (1-\lambda) T(\mathbf{v})$$

for all $\mathbf{u}, \mathbf{v} \in C_V$ and $\lambda \in (0,1)$.

Affine maps $T\colon C_V \to C_W$ have the properties that

(1) images of convex sets are convex;

(2) pre-images of convex sets are convex;

(3) for any $\mathbf{w} \in \mathrm{extr}(C_W)$, for any $\mathbf{v} \in T^{-1}(\{\mathbf{w}\})$,

$$\mathbf{v} \in \mathrm{extr}(C_V) \iff \mathbf{v} \in \mathrm{extr}(T^{-1}(\{\mathbf{w}\})).$$

Structure of classical invariant measures

$(X, \mathcal{X})$ – measurable space.
Vector space $\mathbb{R}^{\mathcal{X}} = \{\text{functions } \mathcal{X} \to \mathbb{R}\}$.

The set

$$\Pr(X, \mathcal{X}) = \{\text{probability measures on } (X, \mathcal{X})\}$$

is a convex subset of $\mathbb{R}^{\mathcal{X}}$:

$$\mu_1, \mu_2 \in \Pr(X, \mathcal{X}), \lambda \in (0,1) \implies \lambda\mu_1 + (1-\lambda)\mu_2 \in \Pr(X, \mathcal{X}).$$

Given $(\mathcal{X}, \mathcal{X})$-measurable map $f\colon X \to X$, the set

$$\mathrm{Inv}(f) := \{f\text{-invariant measures}\} \subset \Pr(X, \mathcal{X})$$

is convex.

Structure of classical invariant measures

Proposition

The set of f-ergodic measures is precisely $\mathrm{extr}(\mathrm{Inv}(f))$.

"extreme⇒ergodic" is the easy direction (by contrapositive):

- suppose $f^{-1}(A) = A$ and $\mu(A) \in (0, 1)$;
- the measures $\mu_1 = \mu(\,\cdot\,|A)$ and $\mu_2 = \mu(\,\cdot\,|X \setminus A)$ are distinct f-invariant measures;
- taking $\lambda = \mu(A)$, we have $\mu = \lambda\mu_1 + (1 - \lambda)\mu_2$.

"ergodic⇒extreme" is harder (but not extremely difficult).

Example

Let $X = \{1, 2, 3, 4\}$ (with $\mathcal{X}$ consisting of all subsets of X).
Let $f = (2\ \ 4)$.

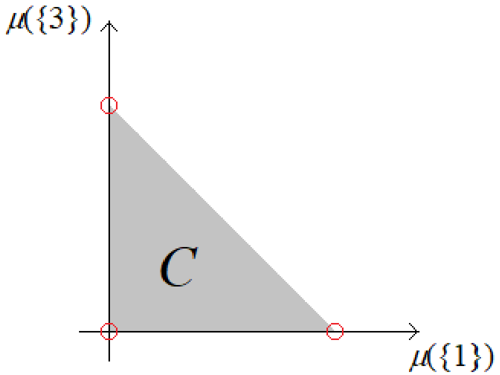
$$\begin{tikzcd} 1 \arrow[loop above] & 2 \arrow[rr, bend right, <->] & 3 \arrow[loop above] & 4 \end{tikzcd}$$

$\mu \in \mathrm{Pr}(X, \mathcal{X})$ is f-invariant if and only if $\mu(\{2\}) = \mu(\{4\})$.
Therefore $\mathrm{Inv}(f)$ is a closed solid triangle:

$$\begin{aligned} \mathbb{R}^{\mathcal{X}} &\to \mathbb{R}^2 \\ \mu &\mapsto (\, \mu(\{1\}) \,,\, \mu(\{3\}) \,) \end{aligned}$$

identifies $\mathrm{Inv}(f)$ with the closed solid triangle C with corners $(0, 0)$, $(0, 1)$, $(1, 0)$.

Example



The f-ergodic measures are precisely the three corner points of the triangle $\mathrm{Inv}(f)$:

$$\delta_1, \quad \delta_3, \quad \tfrac{1}{2}(\delta_2 + \delta_4)$$

Corollary for "skew-product" maps

Let $(X, \mathcal{X})$ be a measurable space,
let $\mathcal{Y}$ be a sub-σ-algebra of $\mathcal{X}$,
and let $f \colon X \to X$ be both $(\mathcal{X}, \mathcal{X})$- and $(\mathcal{Y}, \mathcal{Y})$-measurable.

[This is the most general form of a ***skew-product map****:*

- *models joint evolution of a "driving process" and a "driven process";*
- *hypothetical example: human activity and the climate;*
- *the sub-σ-algebra $\mathcal{Y}$ represents just the driving process.]*

Fix $\nu \in \Pr(X, \mathcal{Y})$ that is ergodic w.r.t. f as a map on $(X, \mathcal{Y})$.

[I.e. the driving is an ergodic process.]

Corollary for "skew-product" maps

Let $C = \{\mu \in \mathrm{Pr}(X, \mathcal{X}) \,:\, \mu \text{ is } f\text{-invariant and } \mu|_{\mathcal{Y}} = \nu\}$.

[I.e. C is the set of all those invariant measures of the joint driver-and-driven dynamics that are compatible with the pre-fixed ergodic measure ν of the driving dynamics.]

Corollary

A measure $\mu \in C$ is f-ergodic if and only if $\mu \in \mathrm{extr}(C)$.

Pf. Immediate by property (3) of affine maps, since the map

$$\begin{aligned} \mathrm{Inv}_{(X,\mathcal{X})}(f) &\to \mathrm{Inv}_{(X,\mathcal{Y})}(f) \\ \mu &\mapsto \mu|_{\mathcal{Y}} \end{aligned}$$

is affine.

My original motivation behind RIMs

Trajectories of dynamical systems perturbed by stationary white noise can be described by *"Markov transition probabilities"* (a "non-fuzzy version", as opposed to the "fuzzy version" given by Markov operators).

One can develop **ergodic theory of Markov transition probabilities**, analogous to the ergodic theory of maps (with "stationary measures" in place of "invariant measures").

In [Crauel 1991, Hairer 2005], stationary measures of systems driven by "*coloured* noise" have been introduced.

- $\rightarrow$ Basic idea: consider joint Markov transition probabilities of the driving noise and the driven system (a "stochastic version" of the skew-product setup).

My original motivation behind RIMs

My aim: abstractly formalise this within a general framework of "filtered random dynamical systems" [Arnold 1998]

$\longrightarrow$ In my formalism, the non-fuzzy Markov transition probabilities get replaced by "Koopman" operators associated with RIMs.

My original motivation behind RIMs

“ergodic$\Leftrightarrow$extreme”, and its corollary for skew-product maps, easily extend to non-fuzzy Markov transition probabilities.

Hence ergodic measures of coloured-noise-driven systems are the extreme points of the set of stationary measures.

$\rightarrow$ It is important that my “fuzzied” abstract formalism still maintains this property!

So I was hoping that “ergodic$\Leftrightarrow$extreme” and its corollary would also easily extend to restrictedly invariant measures . . .

Extreme implies ergodic

$(X, \mathcal{X})$, sub-σ-algebra $\tilde{\mathcal{X}} \subset \mathcal{X}$,
$(\mathcal{X}, \tilde{\mathcal{X}})$-measurable map $f\colon X \to X$

Let $\mathrm{Inv}_{\tilde{\mathcal{X}}}(f) \subset \mathrm{Pr}(X, \mathcal{X})$ be the set of all $\tilde{\mathcal{X}}$-restrictedly f-invariant measures. Note that $\mathrm{Inv}_{\tilde{\mathcal{X}}}(f)$ is convex.

Proposition

Let $\mu \in \mathrm{Inv}_{\tilde{\mathcal{X}}}(f)$. Suppose there exists a σ-algebra $\mathcal{D}$ with

$$\tilde{\mathcal{X}} \subset \mathcal{D} \subset \mathcal{X}$$

such that $\mu|_{\mathcal{D}}$ is an extreme point of $\{\mu'|_{\mathcal{D}} : \mu' \in \mathrm{Inv}_{\tilde{\mathcal{X}}}(f)\}$. Then μ is $\tilde{\mathcal{X}}$-restrictedly f-ergodic.

(Proof essentially identical to the classical case.)

What about a converse? (Ergodic⇒extreme?)

Let $X = \{1, 2, 3, 4\}$, with $\mathcal{X}$ consisting of all subsets of X. Let

$$f(x) = x + 1 \bmod 4.$$

(So the only f-invariant measure is $\frac{1}{4}(\delta_1 + \delta_2 + \delta_3 + \delta_4)$.)

Let $\tilde{\mathcal{X}} = \{\emptyset, X, \{1, 2\}, \{3, 4\}\}$. So,
$f^{-1}\tilde{\mathcal{X}} = \{\emptyset, X, \{1, 4\}, \{3, 2\}\}$.

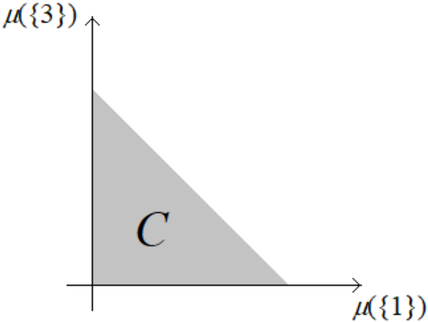
Given $\mu \in \mathrm{Pr}(X, \mathcal{X})$,

- $\mu \in \mathrm{Inv}_{\tilde{\mathcal{X}}}(f)$ if and only if $\mu(\{2\}) = \mu(\{4\})$;
- $\mu \in \mathrm{Inv}_{\tilde{\mathcal{X}}}(f) \setminus \mathrm{Erg}_{\tilde{\mathcal{X}}}(f)$ if and only if

$$\begin{cases} \mu(\{2\}) = \mu(\{4\}) = 0 \\ \mu(\{1\}), \mu(\{3\}) \neq 0. \end{cases}$$

What about a converse? (Ergodic⇒extreme?)

So $\mathrm{Inv}_{\tilde{\mathcal{X}}}(f)$ is once again the triangle:



But $\mathrm{Erg}_{\tilde{\mathcal{X}}}(f) = \mathrm{Inv}_{\tilde{\mathcal{X}}}(f) \setminus \{$open line segment from δ_1 to $\delta_3\}$.

What about a converse? (Ergodic⇒extreme?)

Given any μ in the interior of the triangle $\mathrm{Inv}_{\tilde{\mathcal{X}}}(f)$, there is **no non-trivial σ-algebra $\mathcal{D}$** on X for which $\mu|_{\mathcal{D}}$ is an extreme point of $\{\mu'|_{\mathcal{D}} : \mu' \in \mathrm{Inv}_{\tilde{\mathcal{X}}}(f)\}$.

Conclusion. "ergodic⇒extreme" probably admits no nice extension to the setting of restrictedly invariant measures.

But, for my original goal, it is the corollary for skew-product maps that I need to extend to the RIM setting.

$\rightarrow$ This **does** admit such an extension [that arguably fails on being "nice", but is still enough for what I wanted].

Preliminary: conditional independence

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Recall: $E_1, E_2 \in \mathcal{F}$ are **$\mathbb{P}$-independent** if $\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1)\mathbb{P}(E_2)$.

Fix a sub-σ-algebra $\mathcal{H}$ of $\mathcal{F}$. Two events $E_1, E_2 \in \mathcal{F}$ are **conditionally $\mathbb{P}$-independent given** $\mathcal{H}$ if

$$\mathbb{P}(E_1 \cap E_2|\mathcal{H})(\omega) = \mathbb{P}(E_1|\mathcal{H})(\omega)\,\mathbb{P}(E_2|\mathcal{H})(\omega)$$

"for $\mathbb{P}$-almost all ω". Notation: $E_1 \perp_{\mathbb{P}} E_2 \,|\, \mathcal{H}$.

For two sub-σ-algebras $\mathcal{G}_1, \mathcal{G}_2$ of $\mathcal{F}$, we say that

$$\mathcal{G}_1 \perp_{\mathbb{P}} \mathcal{G}_2 \,|\, \mathcal{H}$$

if for all $E_1 \in \mathcal{G}_1$, $E_2 \in \mathcal{G}_2$, $E_1 \perp_{\mathbb{P}} E_2 \,|\, \mathcal{H}$.

Result for skew-product maps

Let $(X, \mathcal{X})$ be a measurable space, with sub-σ-algebras

$$\tilde{\mathcal{X}} \subset \mathcal{X} \qquad \mathcal{Y} \subset \mathcal{X} \qquad \tilde{\mathcal{Y}} \subset \mathcal{Y} \cap \tilde{\mathcal{X}}.$$

[Illustrative hypothetical model:

- $(X, \mathcal{X})$ *represents global climate and human activity, over a given year*
  – possibly combined with a "noise process" $(\zeta_n)_{n \geq 0}$, *over all time from that year onwards.*
- $\mathcal{Y}$ *represents just human activity over a given year – again combined with* $(\zeta_n)_{n \geq 0}$ *in the "noisy" case.*
- $\tilde{\mathcal{Y}}$ *represents total* CO_2 *emission from human activity in a given year.*
- $\tilde{\mathcal{X}}$ *represents total* CO_2 *emission from human activity in a given year together with average GMT for that year.]*

Result for skew-product maps

$f \colon X \to X$

- Suppose f is $(\mathcal{Y}, \tilde{\mathcal{Y}})$-measurable;
- fix $\nu \in \mathrm{Pr}(X, \mathcal{Y})$ that is $\tilde{\mathcal{Y}}$-restrictedly f-ergodic.
  - $\to$ In practice, f will probably be $(\mathcal{Y}, \mathcal{Y})$-measurable,
  - $\to$ and ν will probably be a classical ergodic measure of f as a map on $(X, \mathcal{Y})$.

[Human activity from year to year is modelled as some ergodic oscillation (e.g. quasiperiodic)
– possibly perturbed by $(\zeta_n)_{n\geq 0}$ modelled as ergodic stationary noise.]

- Suppose f is also $(\sigma(\mathcal{Y} \cup \tilde{\mathcal{X}}), \tilde{\mathcal{X}})$-measurable. $\longleftarrow$strong!

[Next year's average GMT is determined by the combination of this year's average GMT, this year's human activity, and possibly the relevant "noise" (ζ_0, ζ_1).]

Result for skew-product maps

$$C := \{\mu \in \mathrm{Pr}(X, \mathcal{X}) \,:\, \mu \in \mathrm{Inv}_{\tilde{\mathcal{X}}}(f),\ \mu|_{\mathcal{Y}} = \nu, \text{ and } \mathcal{Y} \perp_\mu \tilde{\mathcal{X}} \,|\, \tilde{\mathcal{Y}}\}.$$

[A measure $\mu \in C$ is a joint probability distribution of climate and human behaviour over a year (together with the "noise" $(\zeta_n)_{n\geq 0}$ in the "noisy" case) –

- *compatible with the pre-established ergodic behaviour of human activity and the "noise" $(\zeta_n)_{n\geq 0}$;*
- *yields the same joint probability distribution for next year's average GMT and total CO_2 emission as this year's;*
  (this doesn't automatically guarantee the same distribution across three or more consecutive years!)
- *given the knowledge of a year's total CO_2 emission, all further details of human activity in that year – as well as all details of the "noise" from that year onwards – are probabilistically independent of that year's average GMT.]*

Result for skew-product maps

Lemma

C is convex.

More specifically:

- $C_0 := \{\mu \in \mathrm{Pr}(X, \mathcal{X}) \,:\, \mu|_{\mathcal{Y}} = \nu$ and $\mathcal{Y} \perp_\mu \tilde{\mathcal{X}} \,|\, \tilde{\mathcal{Y}}\}$ is convex (nothing to do with dynamics);
- C is just $C_0 \cap \mathrm{Inv}_{\tilde{\mathcal{X}}}(f)$.

Theorem

A measure $\mu \in C$ is $\tilde{\mathcal{X}}$-restrictedly f-ergodic if and only if $\mu|_{\tilde{\mathcal{X}}}$ is an extreme point of $\{\mu'|_{\tilde{\mathcal{X}}} : \mu' \in C\}$.

We recover the classical "Corollary for skew-product maps" by taking $\tilde{\mathcal{X}} = \mathcal{X}$ and $\tilde{\mathcal{Y}} = \mathcal{Y}$.

Result for skew-product maps

Remark. Another result of classical ergodic theory: distinct f-ergodic measures are mutually singular.

$\rightarrow$ does not generalise to RIMs (clear from earlier example);

$\rightarrow$ but in the above skew-product setting: distinct $\tilde{\mathcal{X}}$-restrictedly f-ergodic measures in C are mutually singular.

Thank you.

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