Some thoughts on stability analysis of relative equilibria and relative periodic orbits in systems with a continous rotational symmetry

(S^1 or $\mathbb{T}$ the circle group, $SO(2)$ special orthogonal group of all rotations in the plane, $U(1)$ unitary group of degree one, all of these are equivalent.)

Reading group 4 Dec 2012

Introduction and notation

Consider $\dot{x} = f(x, p)$ where $x \in \mathbb{R}^n$ and $p \in \mathbb{R}^k$ (1)

A group element $g \in G$ is a symmetry of (1) if for every solution $x(t)$ of (1) $g\,x(t)$ is also a solution.
This implies equivarience $g f(x) = f(gx)$.

Here, we use g to denote an (abstract) element of the group G as well as its action/representation on $\mathbb{R}^n$.

Action/representation of an element of a compact Lie group on $\mathbb{R}^n$ is a $n \times n$ matrix
$g = e^{\Theta T}$, where $\Theta \in \mathbb{R}$ parametrises the transformation and $T_{n\times n}$ satisfies

$T^{*T} = -T$ T is anti-Hermitian or anti self-adjoint

The group orbit of $x \in \mathbb{R}^n$ is the set $Gx = \{gx : g \in G\}$.

A relative equilibrium satisfies $x(0) = g(\tau)\, x(\tau)$ for all τ.
$x(0)$ and $x(\tau)$ are the same up to a shift by the group action $g(\tau)$.
$x(\tau)$ lie on the group orbit of $x(0)$.

A relative periodic orbit satisfies $x(0) = g_p\, x(T_p)$ for some fixed T_p and g_p.
$x(0)$ recurs after some fixed period T but is shifted by the group action $g(T)$. T is the smallest τ such that $x(T)$ lies on the group orbit of $x(0)$.

<u>Exemple:</u> Consider $\begin{cases} \dot{z}_1 = f_1(z_1, z_2) \\ \dot{z}_2 = f_2(z_1, z_2) \end{cases}$; $z_1, z_2 \in \mathbb{C}$

such that f is equiverienb under rotations

$$f(\gamma z_1, \gamma z_2) = \gamma f(z_1, z_2) \qquad \gamma = e^{i\theta} \text{ for } \theta \in (0, 2\pi]$$

Alternatively, $\dot{x} = F(x)$ $x \in \mathbb{R}^4$, $g F(x) = F(gx)$ where

$$g = e^{\theta T} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The system has a continous rotational symmety which implies that in the four-dimensional (z_1, z_2) phase space:

(i) the simplest non-zero solutions with steady $|z_1|$ and $|z_2|$ are typically circular periodic orbits (relative equilibria).

(ii) the simplest solutions with non-steady $|z_1|$ and $|z_2|$ are typically tori (relative periodic orbits)

(A more general statement) Direct time integration is usually straightforward. However, stability analysis of relative equilib. and relative periodic orbits is not!

<u>Goal 1</u>: To facilitate stability analysis we would like to study relative equilibria as isolated equilibria and relative periodic orbits as isolated periodic orbits in some symmetry-reduced system. More specifically, we would like to:

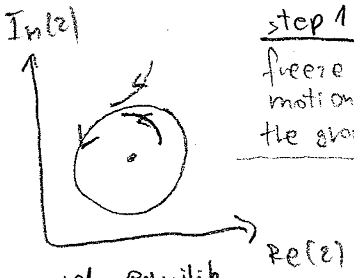
(i) perform local bifurcation analysis.

(ii) compute stable/unstable invariant manifolds of saddles

(ii) perform global (homoclinic, heteroclinic) bifurcation analysis.

<u>Goal 2</u>: If at all possible, the method should be global, that is it should work for the whole of the phase space

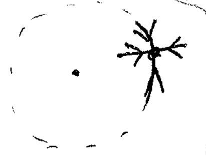
(one may exclude fixed-point subspace that is fixed by the symmetry.



step 1:
freeze the motion along the group orb.

circle of non-isolated neutrally stable equilibria. One can stop here and deal with an extra zero eigenvalue.

step 2:
isolate and stabilise one of the equilib. points



manipulate the equations to remove the symmetry.

leave the equations unchanged but modify/adept the existing bifurcation continuation software (e.g. Auto, Dde-Biftool)

Main concept: represent all points along the group orbit of $x \in \mathbb{R}^n$ by its unique representative $\bar{x}$.

1. Augmented system approach

Consider $\dot{z}_i = f_i(z_1, \ldots, z_n)$ $z_i \in \mathbb{C}$ with rotational symmetry.

To remove a constant drift introduce $\tilde{z} = z \cdot e^{i\omega t}$ =>

$$\dot{\tilde{z}}_i = f_i(\tilde{z}_1, \ldots, \tilde{z}_n) + i\omega \tilde{z}_i$$

To isolate one equilibrium add one extra equation:

$$\dot{\omega} = a\, \mathrm{Re}(z_k) - b\, \mathrm{Im}(z_k)$$

with suitably chosen a, b, z_k

Pros: Often works for relative equilibria and their stability

Cons: - if $z_k = 0$ then it does not work
- the extra equation may mess up the spectrum of the original system (?)
- it is not clear if this works for relative periodic orbits (sometimes it does)

2. Transformation of coordinates

Seek a (nonlinear) transformation of coordinates such that, in the new coordinates, the system does not have the rotational symmetry.

Original system: $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, $f(gx) = g f(x)$

Symmetry-reducing transformation: $y = \Pi(x)$, $\Pi(gx) = \Pi(x)$

Symmetry-reduced system: $\dot{y} = h(y)$, $y \in \mathbb{R}^m$, $m \geq n - N$

For a fixed m, the set $\{y_1, \dots, y_m\}$ is not unique.
A set of y_i's with the smallest m is called a minimal set.

<u>Example:</u> Consider $\dot{z}_i = f_i(z_1, \dots, z_k)$, $z_i \in \mathbb{C}$, $i = 1, \dots, k$,

and $f_i(\gamma z_1, \dots, \gamma z_k) = \gamma f_i(z_1, \dots, z_k)$ for $i = 1, \dots, k$.
The original system is $2k$-dimensional.

<u>choice 1:</u> Pick a reference z_l and use the following $2k-1$ new variables:

$$y_1 = |z_1|^2, \dots, y_k = |z_k|^2, \quad y_{k+1} = \arg(z_1 z_l^*), \dots, y_{2k-1} = \arg(z_k z_l^*).$$

Equations for y_i can be derived using the original system and the chain rule.

Pros: works well for weakly coupled self-sustained amplitude-phase oscill.

Cons: If at least one $z_j = 0$, then at least one y_i is not defined, meaning that this will not work if additional constraints (other symmetries, invariant subspaces) force some z_j to be zero.

Choice 2 : Use the following k^2 new variables:

$$z_i z_j^* \quad \text{for } i = 1, \dots, k \text{ and } j = 1, \dots, k.$$

In this case the new variables form a polynomial expansion in terms of the original variables and are called the invariant generators or the Hilbert basis.

Note that k^2 invariant generators are not independent in the sense that they are related through $k^2 - 2k + N$ (nonlinear) algebraic relations called syzygies:

$$0 = z_i \bar{z}_j \cdot z_l \cdot \bar{z}_m - z_i \bar{z}_j \, \overline{\bar{z}_l z_m} = 0 .$$

(for some symmetries, syzygies may be inequalities!)

Pros: reduced symmetry and no singularities

Cons: - becomes impractical for large k as one has to deal with a large number of ODEs subject to algebraic constraints.

- if the algebraic equations are explicitly eliminated this will introduce singularities in the resulting ODEs.

3. Slices or moving frames approach by Siminos & Cvitanović

This method gives a cute way of finding a transformation of coordinates. In essence, it is very similar to method 2 and could be considered as "choice 3" in method 2.

Preliminaries following the Physica D paper:

At x, there are two components of the vector field: tangent and normal to the group orbit of x. If the tangent component is N-dimensional, then there are N matrices T, and N tangent vectors $t = Tx$.

$(T^*)^T = -T$ implies that t is normal to x.

$$x^T t = x^T Tx = (x^T Tx)^T = x^T (x^T T)^T = x^T T^T x = -x^T Tx = -x^T t = 0.$$

The main idea is that (at least locally) every point along any solution $x(\tau) \in \mathbb{R}^n$ can be mapped to a unique representative $\bar{x}(\tau)$ of all points on its group orbit $G\bar{x}(\tau)$ (including $x(\tau)$) by a suitable shift $g(\theta)$:

$$\bar{x}(\tau) = g(\theta)\, x(\tau).$$

---

step 1: Choose a slice fixing point $\bar{x}' \in \mathbb{R}^n$.

step 2: Define a slice $\bar{M}$ such that $t' = T\bar{x}'$ is orthogonal to $\bar{M}$.

Slice fixing condition: $\underbrace{(\bar{x} - \bar{x}')^T}_{\text{vectors orthogonal to } t'} t' = 0$ ↘ tangent vector at $\bar{x}'$

$$\bar{M} = \{ \bar{x} : (\bar{x} - \bar{x}')^T t' = 0 \}$$

Step 3: Given some $x(\tau)$, calculate the angle θ such that $g(\theta)$ maps $x(\tau)$ onto $\bar{x}(\tau) \in \bar{M}$.

Using slice fixing condition we get

$$\bar{x}^T t' - \underbrace{\bar{x}'t'}_{0} = \bar{x}^T t' = 0 \qquad \text{or} \qquad \boxed{\Big(g(\theta)\underbrace{x(\tau)}_{\text{given}}\Big)^T \underbrace{T\bar{x}'}_{\substack{\text{a vector given by the choice of}\\ \text{of the slice fixing point } \bar{x}'}} = 0}$$

need to solve for θ

We end up with the following problem:

$$\left.\begin{aligned} &\big(g(\theta)\,x(\tau)\big)^T T\bar{x}' = 0 \;, \text{ which gives } \theta(x) \\ &\bar{x}(\tau) = e^{\theta(x)T} x(\tau) \end{aligned}\right\} \quad n-N \text{ equations relating } \bar{x} \text{ to } x.$$

Finally, the equations of motion for the new coordinates are:

$$\dot{\bar{x}} = \frac{d}{dx}\left(e^{\theta(x)T}x\right)\cdot \dot{x}$$

Pros: Gives invariant coordinates of small dimensionality $n-N$

Cons: - The method is local, meaning that there is no guarantee that group orbits for all $x \in \mathbb{R}^n$ cross the slice $\bar{M}$. This is related to singularities in calculating θ from the slice fixing condition.

- The price for small dimensionality $(n-N)$ are singularities in the equations for $\dot{\bar{x}}$.