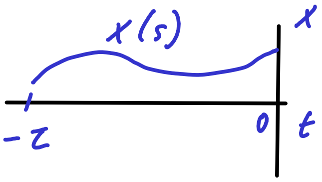
Periodic orbits in differential equations with state-dependent delay

Reminder: consider $\dot{x}(t) = \varphi(x(t), x(t-\tau))$, τ fixed >0

$x(t) \in \mathbb{R}^n$

Initial condition:



continuous function on $x: [-\tau, 0] \to \mathbb{R}^n$

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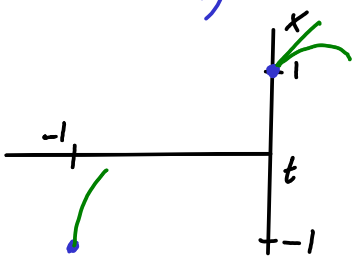
example: $\dot{x}(t) = \cancel{x} - x(t - x(t))$ scalar, try & solve D.E. for $0 \le t \ll 1$



$x(0) = 1$, $x(-1) = -1$

↳ $x(t) = 1 + t$ is solution: $\dot{x}(t) = 1$

$x(t - x(t)) = x(t - 1 - t) = x(-1) = -1$ ✓

When is $x(t) = 1 + t - t^2$ solution?

$\dot{x}(t) = 1 - 2t$

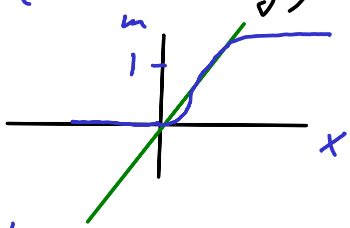
$x(t - x(t)) = x(t - 1 - t + t^2) = x(t^2 - 1) \to x(t^2 - 1) = 2t - 1$

$s = t^2 - 1 \to t = \sqrt{1+s} \to x(s) = 2\sqrt{1+s} - 1$ for $-1 \le s \ll 0$

↳ solution to IVP starting from continuous x is not unique!

This is not a problem of causality ($x(t) < 0$, e.g.)

$\dot{x} = -x(t - m(x(t)))$



Problem is initial slope at $t = -1$

Formulation of IVP: let $x: [-\tau, t] \to \mathbb{R}^n$ be trajectory until now

$f: \underbrace{C^0([-\tau, 0]; \mathbb{R}^n)}_{\text{cont. fcns on } [-\tau, 0]} \to \mathbb{R}^n$ is right-hand side time shift: $t \in \mathbb{R}$, $[\Delta_t x](s) := x(t+s)$

Δ_t maps $x: [(t)-\tau, t] \to \mathbb{R}^n$ to $\Delta_t x : [-\tau, 0] \to \mathbb{R}^n$

Then we can write general DDE as

$$\dot{x}(t) = f(\Delta_t x)$$

in example $\dot{x}(t) = \mu - x(t - x(t))$ $\quad f(x) = \mu - x(-x(0))$

$(f(x) = \varphi(x(0), x(-\tau)))$

$f: U \subseteq C^0([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ continuous, $U = \{x : 0 < x(0) < \tau\}$

But not locally Lipschitz continuous

(in nbh N of x_0 ex. K s.t. $|f(x) - f(y)| \leq K \max_{t \in [-\tau, 0]} |x(t) - y(t)|$ for $x, y \in N$)

$f(x) = \mu - x(-x(0))$!

$\partial f(x) y = x'(-x(0)) y(0) - y(-x(0))$ $\quad \to f: C^k([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$

is k times cont. diff.

also $f: C^{0,1}([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ is locally Lipschitz cont.

Possible phase space: solve $\dot{x} = f(\Delta_t x)$, $\quad x(s) = x_0(s)$ for $s \in [-\tau, 0]$

+ add. cond. on f

$C^{0,1}([-\tau, 0]; \mathbb{R}) \to \Delta_t x$ for $t > 0$ depend L-cont. on x_0

$U := \{x \in C^1([-\tau, 0]; \mathbb{R}) : x'(0) = f(x)\}$ $\quad (x'(0) = \mu - x(-x(0)))$

nonlinear submanifold of C^1, codimension $= 1$ (generally n)

condition prevents corner at $t = 0$

$\hookrightarrow$ semiflow $U \to U$, $\Delta_t x$ depends cont. diff. once on x_0

in metric of $U \subseteq C^1$

(H.-O. Walther)

$\to$ principle of linearized stability ok

bifurcations not ok

general: $\quad U = \{x \in C^1([-\tau_{max}, 0]; \mathbb{R}^n) : x'(0) = f(x)\}$

upper bound for state-dependent delay

What about bifurcation analysis?

Find equilibria: easy! Example: $0 = \mu - x(t - x(t))$,

but $x(\cdot)$ is constant $x(t) = x_0$ for all $t \to 0 = \mu - x_0$

In general, equilibria defined as roots of system of n alg. eq.:

constant extension $E_0 : \mathbb{R}^n \to C([-\tau, 0]; \mathbb{R}^n)$ $(E x_0)(t) := x_0$

$\hookrightarrow$ alg. system is $0 = f(E_0 x_0)$ n eqs., n vars (x_0)

$\hookrightarrow$ all static equilibrium bifurcation ok (fold, pitchfork, transcritical, ..)

local stability of equilibria: lin. stability is ok

linearization in $E_0 x_0$: linear DDE with constant delay

in example: $x_0 = \mu \hookrightarrow \dot{x} = -x(t - \mu)$

Eigenvalues λ: $\lambda = -e^{-\mu\lambda}$ stable if $\mu < \frac{\pi}{2}$

$i\omega = -e^{-i\omega\mu}$ loses stability at $\mu = \frac{\pi}{2}, \omega = 1$

Hopf bifurcation?

Similar result for periodic orbits: can be found as roots of system of algebraic eqs.

More precisely: consider periodic BVPs

$C^k(S, \mathbb{R}^n) := \{x : [-\pi, \pi] \to \mathbb{R}^n, k \text{ times cont. diff}, x^{(j)}(-\pi) = x^{(j)}(\pi)\}$

S "unit circle" $\|x\|_k := \max\{|x|, |x'|, .., |x^{(k)}|\}$

$x \in C^0(S; \mathbb{R}^n)$ can be extended to $\mathbb{R}$: $x(t) := x(t \bmod [-\pi, \pi])$

general periodic BVP:

nonlinear functional $f : C(S; \mathbb{R}^n) \to \mathbb{R}^n$ $f(x) = -x(-x(0))$

time shift $\Delta_t : C(S; \mathbb{R}^n) \to C(S; \mathbb{R}^n)$ $(\Delta_t x)(s) = x(t+s)$

$\dot{x}(t) = f(\Delta_t x) =: F(x)(t)$, $x \in C^1(S; \mathbb{R}^n)$

$F : C(S; \mathbb{R}^n) \to C(S; \mathbb{R}^n)$

for our example:

$$\left.\begin{aligned} \dot{x}(t) &= \tfrac{1}{\omega}\left[\mu - x(t-\omega x(t))\right] \\ \dot{\omega}(t) &= 0 \\ \dot{\mu}(t) &= 0 \end{aligned}\right\} \dot{x}_{ext}(t) = f_{ext}(\Delta_t x_{ext})$$

$x_{ext} = \begin{pmatrix} x \\ \omega \\ \mu \end{pmatrix} \in C^1(S;\mathbb{R}^3)$, $f_{ext}\begin{pmatrix} x \\ \omega \\ \mu \end{pmatrix} = \begin{pmatrix} \frac{1}{\omega(0)}\left[\mu(0) - x(-\omega(0)x(0))\right] \\ 0 \\ 0 \end{pmatrix}$

$\frac{2\pi}{\omega}$ = Period of orbit

$$F_{ext}\begin{pmatrix} x \\ \omega \\ \mu \end{pmatrix}(t) = \begin{bmatrix} \frac{1}{\omega(t)}\left[\mu(t) - x(t-\omega(t)x(t))\right] \\ 0 \\ 0 \end{bmatrix}$$

Main result:

+ regularity condition on f

$\boxed{\begin{array}{l} \dot{x}(t) = F(x)(t) = f(\Delta_t x) \text{ if and only if } g(p) = 0 \text{ and } x = X(p) \\ \text{where } g: \mathbb{R}^m \to \mathbb{R}^n \text{ is } C^k,\ p \in \mathbb{R}^m \text{ and } X: \mathbb{R}^m \to C^0(S,\mathbb{R}^n) \\ \text{is } C^k. \end{array}}$

- m depends on "local Lipschitz constant" of F
- k depends on "smoothness" of F (in example $k = \infty$)
- equivalence is valid **locally** (in nbh of $x_0 \in C^{1,1}(S;\mathbb{R}^n)$)
- $\Rightarrow$ Hopf bif!

(lack of) regularity of f and $F(x) = f(\Delta_t x)$

example: $f(x) = -x(-x(0))$! (ignore μ, ω) $F(x)(t) = -x(t-x(t))$

$$\partial f(x)y = x'(-x(0))y(0) - y(-x(0))$$

$$\partial F(x)y = -x'(t+x(t))y(t) - y(t+x(t))$$

$\hookrightarrow$ f is C^k only as a map $f: C^k(S;\mathbb{R}^n) \to \mathbb{R}^n$

F is C^k only as $F: C^{\ell+k}(S;\mathbb{R}^n) \to C^\ell(S;\mathbb{R}^n)$

$F: C^\ell \to C^\ell$ is only continuous, not locally Lipschitz cont.

Basic idea for construction of algebraic system

Poincaré map not useful: · still an infinite-dimensional map
· at most C^1

Alternative:

$$\dot{x} = F(x)(t) = f(\Delta_t x) \iff x(t) = x(0) + \int_0^t F(x)(s)\,ds$$

↑ does not map $C(S) \to C(S)$

look at this
$$x(t) = p + \int_0^t F(x)(s) - \underbrace{P_0 F(x)(s)}_{P_0 y \,=\, \text{avg of } y}\,ds \qquad \text{(FP)}$$

↑ $p \in \mathbb{R}^n$ parameter

If $x \mapsto E_0 p + \int_0^t F(x)(s) - P_0 F(x)(s)\,ds$ is contraction $\Rightarrow$ (FP) has unique sol: $X(p)$ $\quad p \in \mathbb{R}^n \quad X(p) \in C(S;\mathbb{R}^n)$

alg. system:
$$P_0 F(X(p)) = 0$$

But: $F(x)(t) = -x(t - \frac{\pi}{2}) \leadsto \int_0^t -x(s - \frac{\pi}{2})\,ds$ has EV 1, Evec $\sin t$

Generalize:

$P_N y$ = proj of $y \in C^k(S;\mathbb{R}^n)$ onto first N Fourier modes

$$(P_N y)_i(t) = \sum_{-N}^{N} \langle b_k, y_i \rangle b_k(t) \qquad \langle x, y \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x\, y\, ds$$

$$b_0 = \tfrac{1}{2}, \quad b_k = \cos(kt), \quad b_{-k} = \sin(kt)$$

$Q_N y = y - P_N y$, $P_0 y$ = average of y

$$R_N : C^k(S;\mathbb{R}^n) \to \mathbb{R}^{(2N+1)n} \qquad R_N y_i := \langle b_k, y_i \rangle_{\substack{k=-N..N \\ i=1..n}}$$

$$P_1(\sin t) = \sin t, \quad R_1(\sin t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$E_N : \mathbb{R}^{(2N+1)\times n} \to C^k(S;\mathbb{R}^n) \qquad (E_N p)(t)_i := \sum_{k=-N}^{N} p_{k,i}\, b_k(t) \quad i \in \{1..n\}$$

$$P_N = E_N R_N$$

FP problem: $x(t) = (E_N p)(t) + \left[Q_N \int_0^t F(x)(s)\, ds \right](t)$ (FPP)

$p \in \mathbb{R}^{(2N+1)n}$

$\| Q_N \int_0^t y(s)\, ds \|_\kappa < C \frac{\log N}{N} \| y \|_\kappa$

small

is parametric fixed point problem (par p)

if F is loc. Lipschitz continuous $\|F(x) - F(y)\|_0 \le K \|x-y\|_0$ for all $x, y \in B = \mathcal{N}(x_0)$ and $p \in \operatorname{int} B$ $\rightarrow$ (FPP) has unique solution $x = X(p)$

algebraic system for (p, c): $0 = P_0 F(X(p))$ ← avg of time derivative

$c = E_N p - P_N \int_0^t F(X(p))(s)\, ds$ $\in \operatorname{Im} P_N$

c is constant in $\mathbb{R}^n$. $n(2N+2)$ vars, $n(2N+2)$ eqs

restricted loc. Lipschitz continuity & differentiability

$f(x) = -x(-x(0))$

$\partial f(x) y = x'(-x(0)) y(0) - y(x(0))$ ← does not depend on y' !

$f: C \to \mathbb{R}^n$

$\partial f: C^1 \times C^0 \to \mathbb{R}^n$ continuous

$\lim_{\|y\|_1 \to 0} \frac{|f(x+y) - f(x) - \partial f(x) y|}{\|y\|_1} = 0$ for all $x, y \in C^1$

restricted cont. differentiability

implies: $|f(x) - f(y)| \le K \|x-y\|_0$ for all x, y in suff. small Nbh $\mathcal{N}(x_0) \subseteq C^1$.

also: for all x, y in suff. small Nbh $\mathcal{N}(x_0) \subseteq C^{0,1}$

$\|F(x) - F(y)\|_0 \le K \|x-y\|_0$ for all x, y in sufficiently small Nbh $\mathcal{N}(x_0) \subseteq C^{0,1}$ Lipschitz cont. fcn.

closed balls in $C^{0,1}$ are complete w.r.t. $\|\cdot\|_0$-norm.

↳ we can apply BCMP to $B \subseteq C^{0,1}$ with norm $\|x\|_0 = \max_{t \in S} |x(t)|$