Topological conjugacy of i.i.d. iteration of random circle homeomorphisms (parts 1 and 2)

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<u>Part 1: Thursday 10th October</u>

1 Classical (i.e. non-random) setting

1.1 Set-theoretic dynamical systems

A *set-theoretic dynamical system* (X, f) consists of a set X and a function $f: X \to X$. Given $x \in X$, we define its *orbit* $(f^n(x))_{n\geq 0}$ – heuristically:

- X is the set of possible states of some process (hence we call X the **state space**);
- f is the rule specifying how the process proceeds from its current state to its next state;
- x is an "initial condition" for the process.

The **dynamics** of the dynamical system f is a "soft" term referring to the behaviour of the set of orbits.

We now want a notion of what it means for two set-theoretic dynamical systems $(X, f: X \to X)$ and $(Y, g: Y \to Y)$ to be *the same dynamical system after re-labelling the elements of X by the elements of Y*; in other words, we want a notion of <u>isomorphism</u> for set-theoretic dynamical systems. This is provided by the following:

Definition. Two set-theoretic dynamical systems (X, f) and (Y, g) are **conjugate** if there exists a bijection $h: X \to Y$ such that

$$f = h^{-1} \circ g \circ h.$$

In other words, performing f on X is the same as first translating from X to Y via h, then performing g on Y, and then translating back from Y to X via the inverse of h.

1.2 Topological dynamical systems

Often we do not want to consider the set of states of a process as completely disjointed, but rather as having some notion of what it means for a sequence of states to get arbitrarily close to another state. Hence we would want to equip X with a topology:

A *topological dynamical system* (X, f) consists of a topological space X and a continuous map $f\colon X \to X$. The notion of isomorphism for topological dynamical systems is then as follows:

Definition. Two topological dynamical systems (X, f) and (Y, g) are **topologically conjugate** if there exists a homeomorphism $h\colon X \to Y$ such that

$$f \;=\; h^{-1} \circ g \circ h.$$

2 Random setting

2.1 Random maps

So far, we have considered the situation that the rule specifying how to proceed from the current state to the next state is *deterministic*; we now consider the case that this rule incorporates some influence from some *noise*.

Fix a probability space $(I, \mathcal{I}, \nu)$, which will represent the noise space. (We don't call it $(\Omega, \mathcal{F}, \mathbb{P})$ as that will come later.)

A **random map** on a topological space X is an I-indexed family $(f_\alpha)_{\alpha \in I}$ of continuous maps $f_\alpha\colon X \to X$ such that the map $(\alpha, x) \mapsto f_\alpha(x)$ is measurable (where X is equipped with the Borel σ-algebra).

What this means is that our self-map of X now depends on some parameter α that is realised randomly according to the probability distribution ν.

2.2 Dynamics of a random map

The dynamics of a dynamical system (X, f) was defined essentially as the behaviour arising from iterating the map f. For our purposes here, the dynamics of a random map will analogously be defined as the behaviour arising from **iterating the process of selecting a random α independently of all previously selected α's and applying the associated map f_α**. We formalise this as follows:

Let $(\Omega, \mathcal{F}, \mathbb{P}) := (I^{\mathbb{Z}}, \mathcal{I}^{\otimes\mathbb{Z}}, \nu^{\otimes\mathbb{Z}})$. So Ω is the space of all bi-infinite sequences $(\alpha_i)_{i \in \mathbb{Z}}$ of α-values, where the probability measure $\mathbb{P}$ corresponds to each coordinate α_i having probability distribution ν independently of all the other coordinates. Now in the classical deterministic setup, we may regard $f^{n-m}\colon X \to X$ as being the map taking the state at time m to the state at time n, for any $m, n \in \mathbb{Z}$ with $n \geq m$; analogously in the random setting, for each $\omega = (\alpha_i)_{i \in \mathbb{Z}} \in \Omega$, the map from the state at time $m \in \mathbb{Z}$ to the state at time $n \geq m$ is given by

$$f_{\alpha_{n-1}} \circ \ldots \circ f_{\alpha_m}.$$

In particular, f_{α_0} is the map sending the "current state" – i.e. the state at time 0 – onto the next state.

2.3 Deterministic conjugacy of random maps

We still fix the probability space $(I, \mathcal{I}, \nu)$. Suppose we have a random map $(f_\alpha)_{\alpha \in I}$ on X and a random map $(g_\alpha)_{\alpha \in I}$ on Y? What would it mean for these two random maps to be isomorphic? The answer is given by the following definition:

Definition. Random maps $(f_\alpha)_{\alpha\in I}$ on X and $(g_\alpha)_{\alpha\in I}$ on Y are **deterministically topologically conjugate** if there exists a homeomorphism $h\colon X \to Y$ such that for every $\alpha \in I$,

$$f_\alpha \;=\; h^{-1} \circ g_\alpha \circ h.$$

It is **"very difficult"** for two random maps to be deterministically topologically conjugate – this is a much more degenerate scenario than for two classical topological dynamical systems to be topologically conjugate (assuming $(I, \mathcal{I}, \nu)$ is non-trivial).

So the question arises as to whether we can find a weaker and "more realistic" way to extend the notion of topological conjugacy from the classical setting to the random setting.

> **General principle:** It is easier for two objects to be isomorphic when they are equipped with a weaker structure.

For example, the circle and an interval are *not* isomorphic as topological spaces (i.e. they are not homeomorphic); but if we remove the topological structure and just consider them as sets, then the circle and an interval *are* isomorphic as sets.

So likewise, we will arrive at our definition of conjugacy by, crudely speaking, "weakening the structure of a random map" and then taking the isomorphism of the result.

If we now simply gave the definition, then it would likely seem like it had been pulled out of nowhere. So instead, we will take a detour to describe a concept analogous to how we shall "weaken the structure" of a random map, and with this analogy in mind we will formulate our definition of conjugacy for random maps.

2.4 Analogy from physics

Fix a 3D coordinate system – say, the origin is a particular corner of the floor of the room you are in, with an x-axis, y-axis and z-axis extending from that corner along the boundaries of the two walls that meet there. Ignoring units of distance, this coordinate system provides an identification of 3D space with $\mathbb{R}^3$. Now suppose we have a particle in the room, whose position within this coordinate system as a function of time is given by $\zeta_1(t) \in \mathbb{R}^3$. The evolution of $\zeta_1(t)$ is governed by **Newton's laws**, which can loosely speaking be regarded – for the purpose of our analogy – as a dynamical system specifying the evolution of the position of the particle.

Now suppose we consider the same particle, in a different set of coordinates where the origin is a corner of the ceiling of some room in another building. Suppose the position of the particle in this new set of coordinates is given by $\zeta_2(t)$. The path $\zeta_2(\cdot)$ is a different path in $\mathbb{R}^3$ from the path $\zeta_1(\cdot)$, and yet they are describing exactly the same object, namely the motion of the particle as governed by Newton's laws. The fact that they are describing the same object is manifested through the existence of an **isometry** $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(\zeta_1(t)) = \zeta_2(t)$ for all t. In other words, the two paths are the same path after transforming via h. Here, the isometry h will be analogous to the homeomorphism appearing in the definition of deterministic topological conjugacy.

Now suppose we have a third coordinate system, defined relative to someone who is driving a car along the road outside the building you are in. In this coordinate system, say the position of the particle we have been considering is given by $\tilde{\zeta}(t) \in \mathbb{R}^3$. Note that $\tilde{\zeta}(t)$ is describing exactly the same object as $\zeta_1(t)$ and $\zeta_2(t)$, and yet *there is no isometry of* $\mathbb{R}^3$ *that maps* $\zeta_1(\cdot)$ *or* $\zeta_2(\cdot)$ *onto* $\tilde{\zeta}(\cdot)$. If we want a "weaker notion of isometry" that takes into account relative motion among different reference frames:

- First, regard the motion $t \mapsto \zeta(t)$ of a particle as a motion through **spacetime**, $t \mapsto (t, \zeta(t))$.

- Next, observe that the identification of spacetime as the Cartesian product of time$\cong \mathbb{R}$ and space$\cong \mathbb{R}^3$ makes reference to the coordinate system via which space is identified with $\mathbb{R}^3$. The particular choice of coordinate system is a **stronger structure on spacetime than is necessary to describe spacetime**. Now remove this additional structure by regarding spacetime as the union of disjoint copies of 3-dimensional space associated to each moment in time,

$$\text{spacetime} \cong \bigcup_{t\in\mathbb{R}} \{t\} \times X[t]$$

  where $X[t]$ is isometric to $\mathbb{R}^3$. So **we have removed the ability to say whether two points in spacetime have the same spatial coordinates, except in the case that they have the same temporal coordinate**.

- Motivated by this weaker structure on spacetime, a "weak isometry" of $\mathbb{R} \times \mathbb{R}^3$ is a map $H \colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$ such that H maps $\{t\} \times \mathbb{R}^3$ onto $\{t\} \times \mathbb{R}^3$ and, letting $h_t \colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by $H(t, x) = (t, h_t(x))$, h_t is an isometry of $\mathbb{R}^3$.

With this approach, there is a weak isometry that maps the path $(t, \zeta_1(t))$ onto the path $(t, \tilde{\zeta}(t))$.

Now in a given coordinate system, Newton's laws can be regarded as a dynamical system specifying the motion of particles through space; but Newton's laws themselves do not single out any one coordinate system as special.[1] Therefore, working with our weaker structure of spacetime, Newton's laws may be regarded as a dynamical system specifying the motion of particles through spacetime, where the temporal component is always *constant-speed progression through time*. Thus, from the spatial perspective, Newton's laws dictate how the position in $X[t_1]$ of a particle at time t_1 will progress onto the position in $X[t_2]$ of the particle at time t_2.

2.5 Defining topological conjugacy

Heuristically, the "weaker structure" of a random map is as follows:

- A random map $(f_\alpha)_{\alpha\in I}$ on "space" X defines a measurable map Θ on "spacetime" $\Omega \times X$ given by

$$\Theta(\omega, x) \;=\; (\theta\omega, f_{\alpha_0}(x))$$

  where $\omega = (\alpha_i)_{i\in\mathbb{Z}}$ and $\theta\omega = \theta((\alpha_i)_{i\in\mathbb{Z}}) := (\alpha_{i+1})_{i\in\mathbb{Z}}$. Just as Ω is analogous to "time" in the above setting, the shift map $\theta \colon \Omega \to \Omega$ is analogous to "progression through time".

- We weaken the structure of "spacetime" from the Cartesian product $\Omega \times X$ to the disjoint union $\bigcup_{\omega\in\Omega} \{\omega\} \times X[\omega]$ where $X[\omega]$ is homeomorphic to X. But this disjoint union is not "completely disorderly": we still keep the fibres $\{\omega\} \times X[\omega]$ "glued together" by keeping on the weaker structure $\bigcup_{\omega\in\Omega} \{\omega\} \times X[\omega]$ the σ-algebra inherited from the stronger structure $\Omega \times X$ equipped with its natural σ-algebra $\mathcal{F} \otimes \mathcal{B}(X)$.

[1] They do single out as special one equivalence class of coordinate systems under the equivalence relation of zero relative rotation and constant-speed relative translation; but we will ignore this.

- From this point of view, Θ is still a measurable map from $\bigcup_{\omega\in\Omega}\{\omega\} \times X[\omega]$ to itself, with the map f_{α_0} which sends the current state onto the next state being a map from $X[\omega]$ to $X[\theta\omega]$.

Measurable maps on "spacetime" $\bigcup_{\omega\in\Omega}\{\omega\} \times X[\omega]$ whose Ω-component coincides with θ will be identified up to $\mathbb{P}$-almost everywhere equality of the associated "spatial" mapping from $X[\omega]$ to $X[\theta\omega]$.

2.5.1 Rigorous definition of "topological conjugacy" (without "deterministic")

Fix a probability space $(I, \mathcal{I}, \nu)$ and let $(\Omega, \mathcal{F}, \mathbb{P}) := (I^{\mathbb{Z}}, \mathcal{I}^{\otimes\mathbb{Z}}, \nu^{\otimes\mathbb{Z}})$. We define $\theta\colon \Omega \to \Omega$ to be the left-shift map, $\theta((\alpha_i)_{i\in\mathbb{Z}}) := (\alpha_{i+1})_{i\in\mathbb{Z}}$.

Definition. We say that a random map $(f_\alpha)_{\alpha\in I}$ on a topological space X and a random map $(g_\alpha)_{\alpha\in I}$ on a topological space Y **have topologically conjugate dynamics** if there exists a measurably invertible function $H\colon \Omega \times X \to \Omega \times Y$ with $H(\{\omega\} \times X) = \{\omega\} \times Y$ for all $\omega \in \Omega$, such that writing

$$H(\omega, x) \;=\; (\omega, h_\omega(x))$$

we have:

- the map $h_\omega\colon X \to Y$ is a homeomorphism for all $\omega \in \Omega$;
- for $\mathbb{P}$-almost every $\omega = (\alpha_i)_{i\in\mathbb{Z}} \in \Omega$,

$$f_{\alpha_0} \;=\; h_{\theta\omega}^{-1} \circ g_{\alpha_0} \circ h_\omega.$$

The heuristic interpretation is: to apply the mapping f_{α_0} from $X[\omega]$ to $X[\theta\omega]$, we first translate from $X[\omega]$ to $Y[\omega]$ via h_ω, we then apply the mapping g_{α_0} from $Y[\omega]$ to $Y[\theta\omega]$, and we then translate back from $Y[\theta\omega]$ to $X[\theta\omega]$ via the inverse of $h_{\theta\omega}$.

It turns out that taking Ω to be the *two-sided* sequence space $I^{\mathbb{Z}}$ rather than the one-sided sequence space $I^{\mathbb{N}_0}$ is very significant: including the negative-time coordinates in Ω allows much more flexibility in the set of maps h_ω. In fact, using the one-sided sequence space in the above definition makes it barely weaker than deterministic topological conjugacy.

Part 2: Thursday 24th October

3 Dynamics on the circle S^1

3.1 The state space S^1

Last time: a dynamical system is, heuristically, a rule specifying how to proceed from the *current state* $x \in X$ to the *next state* $f(x) \in X$. The set X is called the *state space*, and represents the set of possible states of a process.

We now suppose the set X of possible states is **topologically a circle**; in other words, the process is what we may call a **"cyclic process"** [Ant84]. We will denote this state space X as S^1.

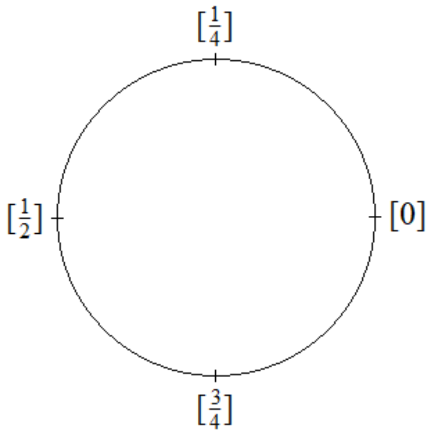
Now there are two possible "directions of travel" along the circular state space S^1. We may wish to equip S^1 with an **orientation**: this is a specification of which of the two directions we consider to be "progressing forward through the cycle" (**anticlockwise**) and which one we consider to be "regressing backward through the cycle" (**clockwise**). So a given topological circle always has two possible orientations; a topological circle equipped with a choice of orientation is called an **oriented topological circle**.

Now all the results that we will present are for random maps on a topological circle S^1, with no extra structure required on the circle. However, "for convenience", we will do the following two things:

(1) we will equip S^1 with an orientation;

(2) we will *calibrate* the oriented space S^1 – heuristically:

- we choose a state in S^1 that we consider to correspond to the "start of a cycle", which we denote $[0]$;
- from $[0]$ we go round the circle assigning to each state in S^1 a number in $[0, 1)$ (written inside square brackets $[\cdot]$) which represents "what proportion through a cycle we are at".



Slightly more formally, our "calibration" is a continuous bijective map

$$[0, 1) \underset{[\cdot]}{\longrightarrow} S^1$$

such that as x increases in $[0, 1)$, $[x]$ moves anticlockwise round S^1. Now observe that as $x \nearrow 1$, $[x] \to [0]$. Therefore, define $[1] = [0]$. Then, by extension (since we can continue going anticlockwise round the circle, and can likewise also go clockwise round the circle indefinitely), define

$$[x + n] \;=\; [x] \qquad \forall\, x \in [0, 1), n \in \mathbb{Z}.$$

So, in algebraic terms, our calibration is an identification of S^1 with the quotient $\mathbb{R}/\mathbb{Z}$ of the additive group $(\mathbb{R}, +)$ by its subgroup $\mathbb{Z}$. Note that by this identification, the sum $[x]+[y] := [x+y]$ of two elements $[x]$ and $[y]$ of S^1 is well-defined.

Note that from a physical perspective, **the choice of orientation is likely to be physically justified, but the specific choice of calibration is, by comparison, somewhat arbitrary**. For example: The set of positions of the Earth relative to the Sun is (for simplistic purposes) an ellipse, which is a topological circle. This is equipped with a very natural choice of orientation – namely, progression along this ellipse corresponds to the direction of the Earth's motion as time moves forward; but our decision to start the year at January 1st is somewhat arbitrary (and not particularly logical).

3.2 Circle maps $f\colon S^1 \to S^1$

If $f\colon S^1 \to S^1$ is a homeomorphism, then either

- as x moves round the circle, $f(x)$ moves round the circle in the *opposite direction* to x; or
- as x moves round the circle, $f(x)$ moves round the circle in the *same direction* as x.

In the latter case, we say that f is **orientation-preserving**; in the former case, we say that f is **orientation-reversing**. Note that if S^1 is regarded as just a topological circle without a pre-specified orientation, one can still determine whether a given homeomorphism $f\colon S^1 \to S^1$ is orientation-preserving or orientation-reversing: either of the two possible orientations on S^1 would still give the same answer. (By contrast, a homeomorphism between two different topological circles cannot be classified as orientation-preserving or orientation-reversing without first specifying orientations on the two topological circles.)

Recall that two continuous maps $f, g\colon S^1 \to S^1$ are **topologically conjugate** if there exists a homeomorphism $h\colon S^1 \to S^1$ such that $f = h^{-1} \circ g \circ h$. Intuitively, this means that *a different choice of calibration of* S^1 *transforms* f *to* g, **except** that here, the re-calibration is allowed to have the opposite orientation – specifically, this corresponds to the case that h is an orientation-reversing. **Since the choice of orientation is often physically motivated, we may wish for our re-calibration to respect the pre-defined orientation on** S^1; namely:

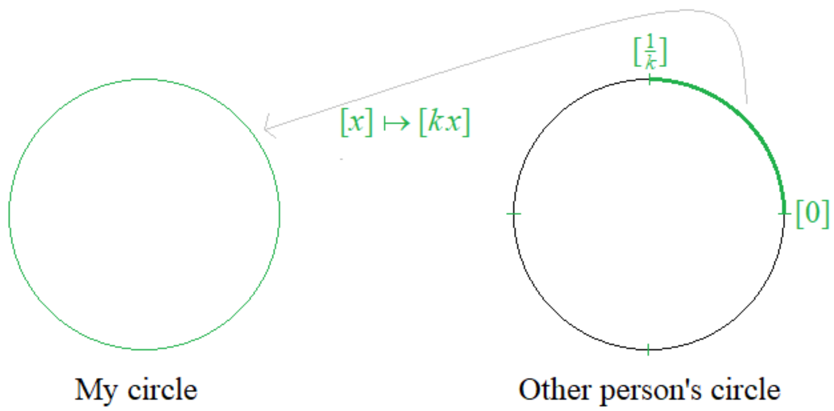
Definition. Two continuous maps $f, g\colon S^1 \to S^1$ are **orientationally conjugate** if there exists an orientation-preserving homeomorphism $h\colon S^1 \to S^1$ such that $f = h^{-1} \circ g \circ h$.

So

- topological conjugacy is the isomorphism of maps on a topological circle;
- orientational conjugacy is the isomorphism of maps on an oriented topological circle.

3.3 Symmetry

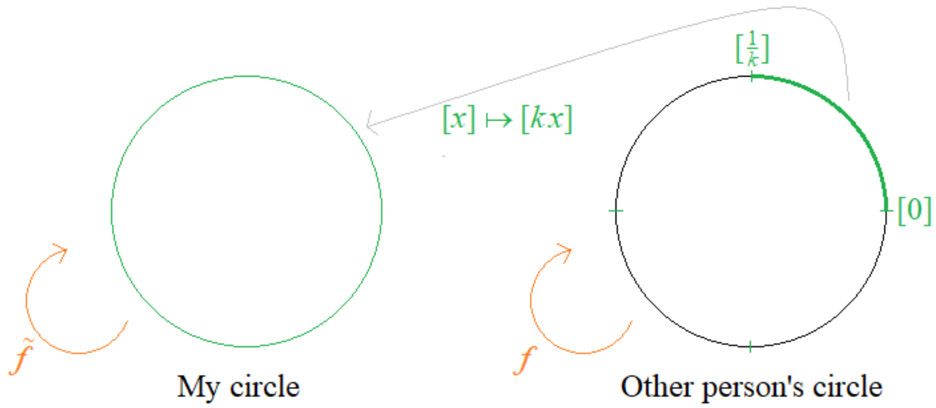
Suppose I am considering the dynamics of some cyclic process with state space S^1; but for some other person, it is every k-th cycle of the process that is significant (where $k \geq 2$). Thus, the "other person's circle S^1" to describe the process would consist of k copies of "my circle S^1"; for simplicity, we can imagine that each arc $\{[x] : \frac{i}{k} \leq x < \frac{i+1}{k}\}$, $i \in \{0, \dots, k-1\}$, of the other person's circle maps onto the whole of my circle via the map $x \mapsto kx$.



Suppose that the evolution of the cyclic process is described by a map $\tilde{f}\colon S^1 \to S^1$, which we will take to be an orientation-preserving homeomorphism; and let us express this map in the form $\tilde{f}([x]) = [\tilde{F}(x)]$ where $\tilde{F}\colon \mathbb{R} \to \mathbb{R}$ is a continuous function. (This is always possible.) Then the same process represented on the other person's circle may be described by an orientation-preserving homeomorphism $f\colon S^1 \to S^1$ of the form

$$f([x]) \;=\; \left[\tfrac{1}{k}\tilde{F}(kx) + \tfrac{i}{k}\right]$$

for some $i \in \{0, \ldots, k-1\}$. A map f of this form is called a *lift of $\tilde{f}$ by order-k rotational symmetry.*



Definition. We say that an orientation-preserving homeomorphism $f\colon S^1 \to S^1$ **has order-k rotational symmetry** if the following equivalent statements hold:

- there is an orientation-preserving homeomorphism $\tilde{f}\colon S^1 \to S^1$ such that f is a lift of $\tilde{f}$ by order-k rotational symmetry;
- $f(x + [\frac{1}{k}]) = f(x) + [\frac{1}{k}]$ for all $x \in S^1$.

Example. For each $k \geq 2$, the map $f([x]) = [x - \frac{1}{10k}\sin(2\pi kx)]$ has order-k rotational symmetry.

Now the above definition was motivated by the heuristic consideration that the other person's calibration of S^1 was the same as mine modulo rotation through $[\frac{1}{k}]$; but the other person could equally well choose any other calibration of their circle that "contains k topological copies of mine". So we now give a purely topological notion of symmetry where, under the above heuristic, other calibrations of the other person's circle are allowed.

Definition. We say that an orientation-preserving homeomorphism $f\colon S^1 \to S^1$ **has order-k topological symmetry** if the following equivalent statements hold:

- f is orientationally conjugate to an orientation-preserving homeomorphism with order-k rotational symmetry;
- there is an orientation-preserving homeomorphism τ with $\inf\{m \geq 1 : \tau^m = \mathrm{id}_{S^1}\} = k$, such that $f \circ \tau = \tau \circ f$.

4 Random circle homeomorphisms

Last time, the noise space was an arbitrary probability space $(I, \mathcal{I}, \nu)$. But in practice, the space of possible noise realisations influencing some process is likely to have some structure by which one can say whether a sequence of possible noise realisation comes "arbitrarily close" to another possible noise realisation. So we will endow I with a topological structure.

More precisely, we fix:

- a **connected** Polish space I, with its Borel σ-algebra $\mathcal{I} = \mathcal{B}(I)$;
- a probability measure ν with full support (i.e. every open subset of I has strictly positive measure).

(A *Polish space* roughly speaking means "a reasonably nice topological space"; more precisely, it is a separable topological space where the topology can be generated by a complete metric.)

From here, we define a **RCH** ("random circle homeomorphism") as an I-indexed family $(f_\alpha)_{\alpha\in I}$ of orientation-preserving homeomorphisms $f_\alpha \colon S^1 \to S^1$ such that the mapping $(\alpha, x) \mapsto f_\alpha(x)$ is continuous and the following *non-degeneracy condition* holds:

(∗) for every non-dense bi-infinite sequence $(x_i)_{i\in\mathbb{Z}}$ in S^1 there exists $\alpha \in I$ and $i \in \mathbb{Z}$ such that $f_\alpha(x_i) \neq x_{i+1}$.

(Don't worry about any intuitive meaning of this; the point is that failing to fulfil this condition is a very degenerate scenario.)

RCHs are a special case of random maps as defined last time, and so we can define deterministic topological conjugacy, as well as topological conjugacy without the word "deterministic", just as last time. Deterministic topological conjugacy between RCHs $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$ is the notion of isomorphism for random maps on a topological circle, meaning intuitively that a re-calibration of S^1 transforms $(f_\alpha)_{\alpha\in I}$ to $(g_\alpha)_{\alpha\in I}$. But once again, we might want to respect a pre-defined orientation on S^1, in which case we need the notion of isomorphism for random maps on an oriented topological circle:

Definition. We say that two RCHs $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$ are **deterministically orientationally conjugate** if there exists an orientation-preserving homeomorphism $h \colon S^1 \to S^1$ such that for all $\alpha \in I$,

$$f_\alpha \;=\; h^{-1} \circ g_\alpha \circ h.$$

This means intuitively that there is a re-calibration of S^1 which keeps the same orientation, by which $(f_\alpha)_{\alpha\in I}$ is transformed to $(g_\alpha)_{\alpha\in I}$.

Now as last time, we let $(\Omega, \mathcal{F}, \mathbb{P}) = (I^{\mathbb{Z}}, \mathcal{B}(I)^{\otimes\mathbb{Z}}, \nu^{\otimes\mathbb{Z}})$.

Lemma. *Suppose we have RCHs $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$, and a function $H \colon \Omega \times S^1 \to \Omega \times S^1$ as in the definition (from last time) of topologically conjugate dynamics. Writing $H(\omega, x) = (\omega, h_\omega(x))$, we have that either*

(i) h_ω is orientation-preserving for $\mathbb{P}$-almost all ω, or

(ii) h_ω is orientation-reversing for $\mathbb{P}$-almost all ω.

Definition. If H as in the definition of topologically conjugate dynamics can be chosen so that (i) holds, then we say that $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$ **have orientationally conjugate dynamics**.

[Ant84] V. A. Antonov. Modeling of processes of cyclic evolution type. Synchronization by a random signal. *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.*, (vyp. 2):67–76, 1984.