Topological conjugacy of i.i.d. iteration of random circle homeomorphisms

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Abstract

We classify random circle homeomorphisms defined over a given connected noise space up to stochastic topological conjugacy of the random dynamical systems that they generate. Here, stochastic conjugacy is understood as a base-space-preserving conjugacy of the naturally arising skew product maps. Some aspects of the classification are quite surprising and counter-intuitive, especially in regards to random homeomorphisms possessing finite-order symmetry.

Part 1: Thursday 10th October

1 Classical (i.e. non-random) setting

1.1 Set-theoretic dynamical systems

A *set-theoretic dynamical system* (X, f) consists of a set X and a function $f\colon X \to X$. Given $x \in X$, we define its *orbit* $(f^n(x))_{n\geq 0}$ – heuristically:

- X is the set of possible states of some process (hence we call X the **state space**);
- f is the rule specifying how the process proceeds from its current state to its next state;
- x is an "initial condition" for the process.

The **dynamics** of the dynamical system f is a "soft" term referring to the behaviour of the set of orbits.

We now want a notion of what it means for two set-theoretic dynamical systems $(X, f\colon X \to X)$ and $(Y, g\colon Y \to Y)$ to be *the same dynamical system after re-labelling the elements of* X *by the elements of* Y; in other words, we want a notion of isomorphism for set-theoretic dynamical systems. This is provided by the following:

Definition. Two set-theoretic dynamical systems (X, f) and (Y, g) are **conjugate** if there exists a bijection $h\colon X \to Y$ such that

$$f \;=\; h^{-1} \circ g \circ h.$$

In other words, performing f on X is the same as first translating from X to Y via h, then performing g on Y, and then translating back from Y to X via the inverse of h.

1.2 Topological dynamical systems

Often we do not want to consider the set of states of a process as completely disjointed, but rather as having some notion of what it means for a sequence of states to get arbitrarily close to another state. Hence we would want to equip X with a topology:

A *topological dynamical system* (X, f) consists of a topological space X and a continuous map $f\colon X \to X$. The notion of isomorphism for topological dynamical systems is then as follows:

Definition. Two topological dynamical systems (X, f) and (Y, g) are **topologically conjugate** if there exists a homeomorphism $h\colon X \to Y$ such that

$$f \;=\; h^{-1} \circ g \circ h.$$

2 Random setting

2.1 Random maps

So far, we have considered the situation that the rule specifying how to proceed from the current state to the next state is *deterministic*; we now consider the case that this rule incorporates some influence from some *noise*.

Fix a probability space $(I, \mathcal{I}, \nu)$, which will represent the noise space. (We don't call it $(\Omega, \mathcal{F}, \mathbb{P})$ as that will come later.)

A **random map** on a topological space X is an I-indexed family $(f_\alpha)_{\alpha \in I}$ of continuous maps $f_\alpha\colon X \to X$ such that the map $(\alpha, x) \mapsto f_\alpha(x)$ is measurable (where X is equipped with the Borel σ-algebra).

What this means is that our self-map of X now depends on some parameter α that is realised randomly according to the probability distribution ν.

2.2 Dynamics of a random map

The dynamics of a dynamical system (X, f) was defined essentially as the behaviour arising from iterating the map f. For our purposes here, the dynamics of a random map will analogously be defined as the behaviour arising from **iterating the process of selecting a random α independently of all previously selected α's and applying the associated map** f_α. We formalise this as follows:

Let $(\Omega, \mathcal{F}, \mathbb{P}) := (I^{\mathbb{Z}}, \mathcal{I}^{\otimes\mathbb{Z}}, \nu^{\otimes\mathbb{Z}})$. So Ω is the space of all bi-infinite sequences $(\alpha_i)_{i\in\mathbb{Z}}$ of α-values, where the probability measure $\mathbb{P}$ corresponds to each coordinate α_i having probability distribution ν independently of all the other coordinates. Now in the classical deterministic setup, we may regard $f^{n-m}\colon X \to X$ as being the map taking the state at time m to the state at time n, for any $m, n \in \mathbb{Z}$ with $n \geq m$; analogously in the random setting, for each $\omega = (\alpha_i)_{i\in\mathbb{Z}} \in \Omega$, the map from the state at time $m \in \mathbb{Z}$ to the state at time $n \geq m$ is given by

$$f_{\alpha_{n-1}} \circ \ldots \circ f_{\alpha_m}.$$

In particular, f_{α_0} is the map sending the "current state" – i.e. the state at time 0 – onto the next state.

2.3 Deterministic conjugacy of random maps

We still fix the probability space $(I, \mathcal{I}, \nu)$. Suppose we have a random map $(f_\alpha)_{\alpha\in I}$ on X and a random map $(g_\alpha)_{\alpha\in I}$ on Y? What would it mean for these two random maps to be isomorphic? The answer is given by the following definition:

Definition. Random maps $(f_\alpha)_{\alpha\in I}$ on X and $(g_\alpha)_{\alpha\in I}$ on Y are **deterministically topologically conjugate** if there exists a homeomorphism $h\colon X \to Y$ such that for every $\alpha \in I$,

$$f_\alpha \;=\; h^{-1} \circ g_\alpha \circ h.$$

It is **"very difficult"** for two random maps to be deterministically topologically conjugate – this is a much more degenerate scenario than for two classical topological dynamical systems to be topologically conjugate (assuming $(I, \mathcal{I}, \nu)$ is non-trivial).

So the question arises as to whether we can find a weaker and "more realistic" way to extend the notion of topological conjugacy from the classical setting to the random setting.

> **General principle:** It is easier for two objects to be isomorphic when they are equipped with a weaker structure.

For example, the circle and an interval are *not* isomorphic as topological spaces (i.e. they are not homeomorphic); but if we remove the topological structure and just consider them as sets, then the circle and an interval *are* isomorphic as sets.

So likewise, we will arrive at our definition of conjugacy by, crudely speaking, "weakening the structure of a random map" and then taking the isomorphism of the result.

If we now simply gave the definition, then it would likely seem like it had been pulled out of nowhere. So instead, we will take a detour to describe a concept analogous to how we shall "weaken the structure" of a random map, and with this analogy in mind we will formulate our definition of conjugacy for random maps.

2.4 Analogy from physics

Fix a 3D coordinate system – say, the origin is a particular corner of the floor of the room you are in, with an x-axis, y-axis and z-axis extending from that corner along the boundaries of the two walls that meet there. Ignoring units of distance, this coordinate system provides an identification of 3D space with $\mathbb{R}^3$. Now suppose we have a particle in the room, whose position within this coordinate system as a function of time is given by $\zeta_1(t) \in \mathbb{R}^3$. The evolution of $\zeta_1(t)$ is governed by **Newton's laws**, which can loosely speaking be regarded – for the purpose of our analogy – as a dynamical system specifying the evolution of the position of the particle.

Now suppose we consider the same particle, in a different set of coordinates where the origin is a corner of the ceiling of some room in another building. Suppose the position of the particle in this new set of coordinates is given by $\zeta_2(t)$. The path $\zeta_2(\cdot)$ is a different path in $\mathbb{R}^3$ from the path $\zeta_1(\cdot)$, and yet they are describing exactly the same object, namely the motion of the particle as governed by Newton's laws. The fact that they are describing the same object is manifested through the existence of an **isometry** $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(\zeta_1(t)) = \zeta_2(t)$ for all t. In other words, the two paths are the same path after transforming via h. Here, the isometry h will be analogous to the homeomorphism appearing in the definition of deterministic topological conjugacy.

Now suppose we have a third coordinate system, defined relative to someone who is driving a car along the road outside the building you are in. In this coordinate system, say the position of the particle we have been considering is given by $\tilde{\zeta}(t) \in \mathbb{R}^3$. Note that $\tilde{\zeta}(t)$ is describing exactly the same object as $\zeta_1(t)$ and $\zeta_2(t)$, and yet *there is no isometry of* $\mathbb{R}^3$ *that maps* $\zeta_1(\cdot)$ *or* $\zeta_2(\cdot)$ *onto* $\tilde{\zeta}(\cdot)$. If we want a "weaker notion of isometry" that takes into account relative motion among different reference frames:

- First, regard the motion $t \mapsto \zeta(t)$ of a particle as a motion through **spacetime**, $t \mapsto (t, \zeta(t))$.
- Next, observe that the identification of spacetime as the Cartesian product of time$\cong \mathbb{R}$ and space$\cong \mathbb{R}^3$ makes reference to the coordinate system via which space is identified with $\mathbb{R}^3$. The particular choice of coordinate system is a **stronger structure on spacetime than is necessary to describe spacetime**. Now remove this additional structure by regarding spacetime as the union of disjoint copies of 3-dimensional space associated to each moment in time,
$$\text{spacetime} \cong \bigcup_{t\in\mathbb{R}} \{t\} \times X[t]$$
where $X[t]$ is isometric to $\mathbb{R}^3$. So **we have removed the ability to say whether two points in spacetime have the same spatial coordinates, except in the case that they have the same temporal coordinate**.
- Motivated by this weaker structure on spacetime, a "weak isometry" of $\mathbb{R} \times \mathbb{R}^3$ is a map $H\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$ such that H maps $\{t\} \times \mathbb{R}^3$ onto $\{t\} \times \mathbb{R}^3$ and, letting $h_t\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by $H(t,x) = (t, h_t(x))$, h_t is an isometry of $\mathbb{R}^3$.

With this approach, there is a weak isometry that maps the path $(t, \zeta_1(t))$ onto the path $(t, \tilde{\zeta}(t))$.

Now in a given coordinate system, Newton's laws can be regarded as a dynamical system specifying the motion of particles through space; but Newton's laws themselves do not single out any one coordinate system as special.[1] Therefore, working with our weaker structure of spacetime, Newton's laws may be regarded as a dynamical system specifying the motion of particles through spacetime, where the temporal component is always *constant-speed progression through time.* Thus, from the spatial perspective, Newton's laws dictate how the position in $X[t_1]$ of a particle at time t_1 will progress onto the position in $X[t_2]$ of the particle at time t_2.

[1] They do single out as special one equivalence class of coordinate systems under the equivalence relation of zero relative rotation and constant-speed relative translation; but we will ignore this.

2.5 Defining topological conjugacy

Heuristically, the "weaker structure" of a random map is as follows:

- A random map $(f_\alpha)_{\alpha\in I}$ on "space" X defines a measurable map Θ on "spacetime" $\Omega\times X$ given by
$$\Theta(\omega,x) \;=\; (\theta\omega, f_{\alpha_0}(x))$$
where $\omega=(\alpha_i)_{i\in\mathbb{Z}}$ and $\theta\omega=\theta((\alpha_i)_{i\in\mathbb{Z}}):=(\alpha_{i+1})_{i\in\mathbb{Z}}$. Just as Ω is analogous to "time" in the above setting, the shift map $\theta\colon\Omega\to\Omega$ is analogous to "progression through time".

- We weaken the structure of "spacetime" from the Cartesian product $\Omega\times X$ to the disjoint union $\bigcup_{\omega\in\Omega}\{\omega\}\times X[\omega]$ where $X[\omega]$ is homeomorphic to X. But this disjoint union is not "completely disorderly": we still keep the fibres $\{\omega\}\times X[\omega]$ "glued together" by keeping on the weaker structure $\bigcup_{\omega\in\Omega}\{\omega\}\times X[\omega]$ the σ-algebra inherited from the stronger structure $\Omega\times X$ equipped with its natural σ-algebra $\mathcal{F}\otimes\mathcal{B}(X)$.

- From this point of view, Θ is still a measurable map from $\bigcup_{\omega\in\Omega}\{\omega\}\times X[\omega]$ to itself, with the map f_{α_0} which sends the current state onto the next state being a map from $X[\omega]$ to $X[\theta\omega]$.

Measurable maps on "spacetime" $\bigcup_{\omega\in\Omega}\{\omega\}\times X[\omega]$ whose Ω-component coincides with θ will be identified up to $\mathbb{P}$-almost everywhere equality of the associated "spatial" mapping from $X[\omega]$ to $X[\theta\omega]$.

2.5.1 Rigorous definition of "topological conjugacy" (without "deterministic")

Fix a probability space $(I,\mathcal{I},\nu)$ and let $(\Omega,\mathcal{F},\mathbb{P}):=(I^{\mathbb{Z}},\mathcal{I}^{\otimes\mathbb{Z}},\nu^{\otimes\mathbb{Z}})$. We define $\theta\colon\Omega\to\Omega$ to be the left-shift map, $\theta((\alpha_i)_{i\in\mathbb{Z}}):=(\alpha_{i+1})_{i\in\mathbb{Z}}$.

Definition. We say that a random map $(f_\alpha)_{\alpha\in I}$ on a topological space X and a random map $(g_\alpha)_{\alpha\in I}$ on a topological space Y **have topologically conjugate dynamics** if there exists a measurably invertible function $H\colon\Omega\times X\to\Omega\times Y$ with $H(\{\omega\}\times X)=\{\omega\}\times Y$ for all $\omega\in\Omega$, such that writing
$$H(\omega,x) \;=\; (\omega,h_\omega(x))$$
we have:

- the map $h_\omega\colon X\to Y$ is a homeomorphism for all $\omega\in\Omega$;

- for $\mathbb{P}$-almost every $\omega=(\alpha_i)_{i\in\mathbb{Z}}\in\Omega$,
$$f_{\alpha_0} \;=\; h_{\theta\omega}^{-1}\circ g_{\alpha_0}\circ h_\omega.$$

The heuristic interpretation is: to apply the mapping f_{α_0} from $X[\omega]$ to $X[\theta\omega]$, we first translate from $X[\omega]$ to $Y[\omega]$ via h_ω, we then apply the mapping g_{α_0} from $Y[\omega]$ to $Y[\theta\omega]$, and we then translate back from $Y[\theta\omega]$ to $X[\theta\omega]$ via the inverse of $h_{\theta\omega}$.

It turns out that taking Ω to be the *two-sided* sequence space $I^{\mathbb{Z}}$ rather than the one-sided sequence space $I^{\mathbb{N}_0}$ is very significant: including the negative-time coordinates in Ω allows much more flexibility in the set of maps h_ω. In fact, using the one-sided sequence space in the above definition makes it barely weaker than deterministic topological conjugacy.

Part 2: Thursday 24th October

3 Dynamics on the circle S^1

3.1 The state space S^1

Last time: a dynamical system is, heuristically, a rule specifying how to proceed from the *current state* $x \in X$ to the *next state* $f(x) \in X$. The set X is called the *state space*, and represents the set of possible states of a process.

We now suppose the set X of possible states is **topologically a circle**; in other words, the process is what we may call a **"cyclic process"** [Ant84]. We will denote this state space X as S^1.

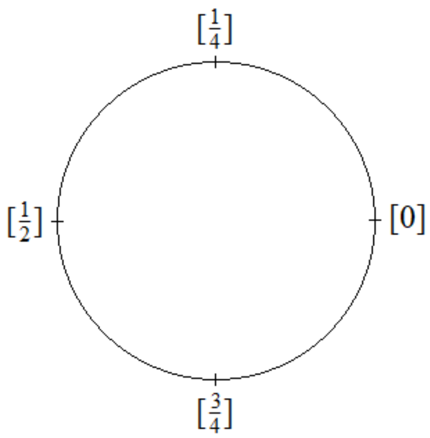
Now there are two possible "directions of travel" along the circular state space S^1. We may wish to equip S^1 with an **orientation**: this is a specification of which of the two directions we consider to be "progressing forward through the cycle" (**anticlockwise**) and which one we consider to be "regressing backward through the cycle" (**clockwise**). So a given topological circle always has two possible orientations; a topological circle equipped with a choice of orientation is called an **oriented topological circle**.

Now all the results that we will present are for random maps on a topological circle S^1, with no extra structure required on the circle. However, "for convenience", we will do the following two things:

(1) we will equip S^1 with an orientation;

(2) we will *calibrate* the oriented space S^1 – heuristically:

- we choose a state in S^1 that we consider to correspond to the "start of a cycle", which we denote [0];
- from [0] we go round the circle assigning to each state in S^1 a number in $[0, 1)$ (written inside square brackets $[\cdot]$) which represents "what proportion through a cycle we are at".



Slightly more formally, our "calibration" is a continuous bijective map

$$[0, 1) \underset{[\cdot]}{\longrightarrow} S^1$$

such that as x increases in $[0,1)$, $[x]$ moves anticlockwise round S^1. Now observe that as $x \nearrow 1$, $[x] \to [0]$. Therefore, define $[1] = [0]$. Then, by extension (since we can continue going anticlockwise round the circle, and can likewise also go clockwise round the circle indefinitely), define

$$[x+n] \;=\; [x] \qquad \forall\, x \in [0,1), n \in \mathbb{Z}.$$

So, in algebraic terms, our calibration is an identification of S^1 with the quotient $\mathbb{R}/\mathbb{Z}$ of the additive group $(\mathbb{R},+)$ by its subgroup $\mathbb{Z}$. Note that by this identification, the sum $[x]+[y] := [x+y]$ of two elements $[x]$ and $[y]$ of S^1 is well-defined.

Note that from a physical perspective, **the choice of orientation is likely to be physically justified, but the specific choice of calibration is, by comparison, somewhat arbitrary**. For example: The set of positions of the Earth relative to the Sun is (for simplistic purposes) an ellipse, which is a topological circle. This is equipped with a very natural choice of orientation – namely, progression along this ellipse corresponds to the direction of the Earth's motion as time moves forward; but our decision to start the year at January 1st is somewhat arbitrary (and not particularly logical).

3.2 Circle maps $f\colon S^1 \to S^1$

If $f\colon S^1 \to S^1$ is a homeomorphism, then either

- as x moves round the circle, $f(x)$ moves round the circle in the *opposite direction* to x; or
- as x moves round the circle, $f(x)$ moves round the circle in the *same direction* as x.

In the latter case, we say that f is **orientation-preserving**; in the former case, we say that f is **orientation-reversing**. Note that if S^1 is regarded as just a topological circle without a pre-specified orientation, one can still determine whether a given homeomorphism $f\colon S^1 \to S^1$ is orientation-preserving or orientation-reversing: either of the two possible orientations on S^1 would still give the same answer. (By contrast, a homeomorphism between two different topological circles cannot be classified as orientation-preserving or orientation-reversing without first specifying orientations on the two topological circles.)

Recall that two continuous maps $f,g\colon S^1 \to S^1$ are **topologically conjugate** if there exists a homeomorphism $h\colon S^1 \to S^1$ such that $f = h^{-1} \circ g \circ h$. Intuitively, this means that *a different choice of calibration of S^1 transforms f to g*, **except** that here, the re-calibration is allowed to have the opposite orientation – specifically, this corresponds to the case that h is an orientation-reversing. **Since the choice of orientation is often physically motivated, we may wish for our re-calibration to respect the pre-defined orientation on S^1**; namely:

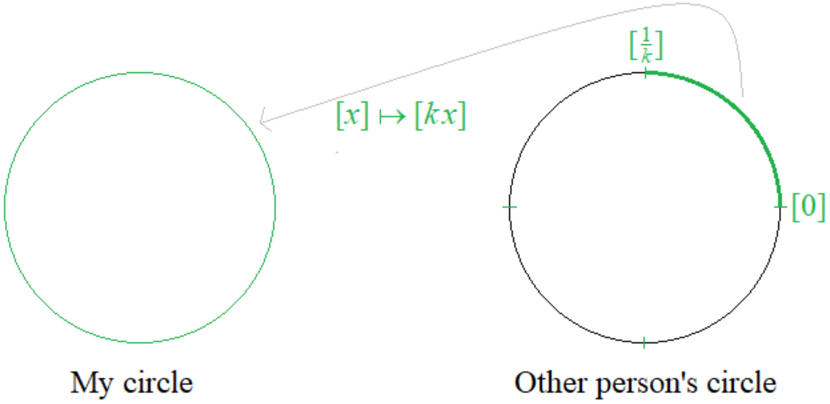
Definition. Two continuous maps $f,g\colon S^1 \to S^1$ are **orientationally conjugate** if there exists an orientation-preserving homeomorphism $h\colon S^1 \to S^1$ such that $f = h^{-1} \circ g \circ h$.

So

- topological conjugacy is the isomorphism of maps on a topological circle;
- orientational conjugacy is the isomorphism of maps on an oriented topological circle.

3.3 Symmetry

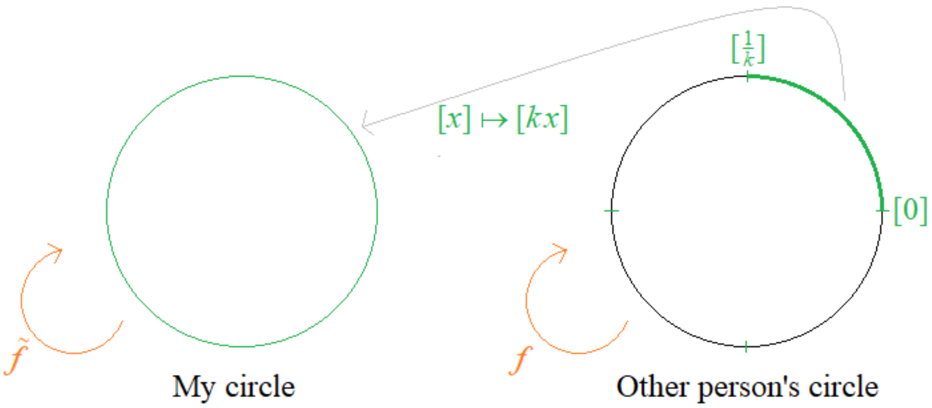
Suppose I am considering the dynamics of some cyclic process with state space S^1; but for some other person, it is every k-th cycle of the process that is significant (where $k \geq 2$). Thus, the "other person's circle S^1" to describe the process would consist of k copies of "my circle S^1"; for simplicity, we can imagine that each arc $\{[x] : \frac{i}{k} \leq x < \frac{i+1}{k}\}$, $i \in \{0, \ldots, k-1\}$, of the other person's circle maps onto the whole of my circle via the map $x \mapsto kx$.



Suppose that the evolution of the cyclic process is described by a map $\tilde{f} \colon S^1 \to S^1$, which we will take to be an orientation-preserving homeomorphism; and let us express this map in the form $\tilde{f}([x]) = [\tilde{F}(x)]$ where $\tilde{F} \colon \mathbb{R} \to \mathbb{R}$ is a continuous function. (This is always possible.) Then the same process represented on the other person's circle may be described by an orientation-preserving homeomorphism $f \colon S^1 \to S^1$ of the form

$$f([x]) \;=\; \left[\tfrac{1}{k}\tilde{F}(kx) + \tfrac{i}{k}\right]$$

for some $i \in \{0, \ldots, k-1\}$. A map f of this form is called a *lift of $\tilde{f}$ by order-k rotational symmetry*.



Definition. We say that an orientation-preserving homeomorphism $f \colon S^1 \to S^1$ **has order-k rotational symmetry** if the following equivalent statements hold:

- there is an orientation-preserving homeomorphism $\tilde{f} \colon S^1 \to S^1$ such that f is a lift of $\tilde{f}$ by order-k rotational symmetry;
- $f(x + [\frac{1}{k}]) = f(x) + [\frac{1}{k}]$ for all $x \in S^1$.

Example. For each $k \geq 2$, the map $f([x]) = [x - \frac{1}{10k}\sin(2\pi kx)]$ has order-k rotational symmetry.

Now the above definition was motivated by the heuristic consideration that the other person's calibration of S^1 was the same as mine modulo rotation through $[\frac{1}{k}]$; but the other person could equally well choose any other calibration of their circle that "contains k topological copies of mine". So we now give a purely topological notion of symmetry where, under the above heuristic, other calibrations of the other person's circle are allowed.

Definition. We say that an orientation-preserving homeomorphism $f\colon S^1 \to S^1$ **has order-k topological symmetry** if the following equivalent statements hold:

- f is orientationally conjugate to an orientation-preserving homeomorphism with order-k rotational symmetry;
- there is an orientation-preserving homeomorphism τ with $\inf\{m \geq 1 : \tau^m = \mathrm{id}_{S^1}\} = k$, such that $f \circ \tau = \tau \circ f$.

4 Random circle homeomorphisms

Last time, the noise space was an arbitrary probability space $(I, \mathcal{I}, \nu)$. But in practice, the space of possible noise realisations influencing some process is likely to have some structure by which one can say whether a sequence of possible noise realisation comes "arbitrarily close" to another possible noise realisation. So we will endow I with a topological structure.

More precisely, we fix:

- a **connected** Polish space I, with its Borel σ-algebra $\mathcal{I} = \mathcal{B}(I)$;
- a probability measure ν with full support (i.e. every open subset of I has strictly positive measure).

(A *Polish space* roughly speaking means "a reasonably nice topological space"; more precisely, it is a separable topological space where the topology can be generated by a complete metric.)

From here, we define a **RCH** ("random circle homeomorphism") as an I-indexed family $(f_\alpha)_{\alpha \in I}$ of orientation-preserving homeomorphisms $f_\alpha\colon S^1 \to S^1$ such that the mapping $(\alpha, x) \mapsto f_\alpha(x)$ is continuous and the following *non-degeneracy condition* holds:

($*$) for every non-dense bi-infinite sequence $(x_i)_{i \in \mathbb{Z}}$ in S^1 there exists $\alpha \in I$ and $i \in \mathbb{Z}$ such that $f_\alpha(x_i) \neq x_{i+1}$.

(Don't worry about any intuitive meaning of this; the point is that failing to fulfil this condition is a very degenerate scenario.)

RCHs are a special case of random maps as defined last time, and so we can define deterministic topological conjugacy, as well as topological conjugacy without the word "deterministic", just as last time. Deterministic topological conjugacy between RCHs $(f_\alpha)_{\alpha \in I}$ and $(g_\alpha)_{\alpha \in I}$ is the notion of isomorphism for random maps on a topological circle, meaning intuitively that a re-calibration of S^1 transforms $(f_\alpha)_{\alpha \in I}$ to $(g_\alpha)_{\alpha \in I}$. But once again, we might want to respect a pre-defined orientation on S^1, in which case we need the notion of isomorphism for random maps on an oriented topological circle:

Definition. We say that two RCHs $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$ are **deterministically orientationally conjugate** if there exists an orientation-preserving homeomorphism $h\colon S^1 \to S^1$ such that for all $\alpha \in I$,

$$f_\alpha \ = \ h^{-1} \circ g_\alpha \circ h.$$

This means intuitively that there is a re-calibration of S^1 which keeps the same orientation, by which $(f_\alpha)_{\alpha\in I}$ is transformed to $(g_\alpha)_{\alpha\in I}$.

Now as last time, we let $(\Omega, \mathcal{F}, \mathbb{P}) = (I^{\mathbb{Z}}, \mathcal{B}(I)^{\otimes\mathbb{Z}}, \nu^{\otimes\mathbb{Z}})$.

Lemma. *Suppose we have RCHs $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$, and a function $H\colon \Omega\times S^1 \to \Omega\times S^1$ as in the definition (from last time) of topologically conjugate dynamics. Writing $H(\omega,x) = (\omega, h_\omega(x))$, we have that either*

(i) h_ω is orientation-preserving for $\mathbb{P}$-almost all ω, or

(ii) h_ω is orientation-reversing for $\mathbb{P}$-almost all ω.

Definition. If H as in the definition of topologically conjugate dynamics can be chosen so that (i) holds, then we say that $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$ **have orientationally conjugate dynamics**.

Part 3: Thursday 31st October

Recap:

As last time, fix a probability space $(I, \mathcal{I}, \nu)$ with

- I a connected Polish space, $\mathcal{I} = \mathcal{B}(I)$,
- ν has full support on I.

With this:

- A *RCH* is a family $(f_\alpha)_{\alpha\in I}$ of orientation-preserving homeomorphisms $f_\alpha\colon S^1 \to S^1$, with continuous dependence on α and fulfilling a very mild nondegeneracy condition.
- Two RCHs $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$ are *deterministically topologically/orientationally conjugate* if there is a homeomorphism h / an orientation-preserving homeomorphism h, such that for every $\alpha \in I$ we have
$$f_\alpha = h^{-1} \circ g_\alpha \circ h.$$
Deterministic conjugacy is very strong; it is an extremely "special" situation for two RCHs to be deterministically topologically conjugate.
- So we defined a weaker notion, "$(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$ *have topologically/orientationally conjugate dynamics*".

5 Basic classification of dynamics of RCHs

Our aim will be to characterise when two RCHs have topologically conjugate dynamics, and to characterise when two RCHs have orientationally conjugate dynamics.

In general, the point of "conjugacy" in dynamical systems is to say that two systems "are dynamically the same after a transformation". So let's present a basic description of the dynamics of RCHs. *It turns out that it's fairly easy to classify the basic dynamical behaviour of RCHs.* (This is part of what will make our "classification via conjugate dynamics" quite surprising!)

First some definitions.

Last time, we discussed symmetry for an orientation-preserving homeomorphism of the circle; we extend the definitions to random circle homeomorphisms in the natural way.

Definition. Given $k \geq 2$, we say that a RCH $(f_\alpha)_{\alpha \in I}$ has

- **order-k rotational symmetry** if f_α has order-k rotational symmetry for all α;
- **order-k topological symmetry** if the following equivalent statements hold:
  - $(f_\alpha)_{\alpha \in I}$ is deterministically orientationally conjugate to a RCH with order-k rotational symmetry;
  - there is an orientation-preserving homeomorphism τ with $\inf\{m \geq 1 : \tau^m = \mathrm{id}_{S^1}\} = k$, such that for every $\alpha \in I$, $f_\alpha \circ \tau = \tau \circ f_\alpha$.

(In the first characterisation of topological symmetry, "orientationally" can equivalently be replaced with "topologically"; so topological symmetry is genuinely a purely topological property.)

Definition. A **rigid-rotation RCH** is a RCH $(f_\alpha)_{\alpha \in I}$ where for each $\alpha \in I$, f_α is simply a rotation map $f_\alpha([x]) = [x + c_\alpha]$ for some $c_\alpha \in [0, 1)$.

Definition. We will say that a RCH is **rigid** if it is is deterministically orientationally equivalent to a rigid-rotation RCH.

(Again, "orientationally" can equivalently be replaced with "topologically".)

Note that *a rigid-rotation RCH has rotational symmetry of every order*; and hence, *a rigid RCH has topological symmetry of every order.*

Definition. A RCH $(f_\alpha)_{\alpha \in I}$ is **minimal** if for every $x \in S^1$ and every non-empty open $U \subset S^1$ there exist $\alpha_0, \ldots, \alpha_{n-1} \in I$ for some $n(x, U) \in \mathbb{N}$, such that $f_{\alpha_{n-1}} \circ \ldots \circ f_{\alpha_0}(x) \in U$.

In other words, minimality means that "you can get from anywhere to anywhere". Due to our non-degeneracy condition, **every rigid RCH is minimal.**

Now recall that we define $(\Omega, \mathcal{F}, \mathbb{P}) = (I^{\mathbb{Z}}, \mathcal{B}(I)^{\otimes\mathbb{Z}}, \nu^{\otimes\mathbb{Z}})$. With this, we can straightforwardly classify the dynamics of i.i.d. iterations of RCHs as follows.

Proposition 1 (based on [Ant84,Mal17])**. (A)** *Let $(f_\alpha)_{\alpha \in I}$ be a **minimal RCH**; then exactly one of the following three statements hold:*

*(a) $(f_\alpha)_{\alpha \in I}$ has **no topological symmetry**, and for $\mathbb{P}$-almost every $(\alpha_i)_{i \in \mathbb{Z}} \in \Omega$ there exists*

- *a "globally repelling point" $r((\alpha_i)_{i \geq 0}) \in S^1$ such that for every compact $G \subset S^1 \setminus \{r\}$, $\mathrm{diam}(f_{\alpha_N} \circ \ldots \circ f_{\alpha_0}(G)) \to 0$ as $N \to \infty$;*
- *a "globally backward-time-repelling point" $a((\alpha_i)_{i<0}) \in S^1$ such that for every compact $G \subset S^1 \setminus \{a\}$, $\mathrm{diam}((f_{\alpha_{-1}} \circ \ldots \circ f_{\alpha_{-N}})^{-1}(G)) \to 0$ as $N \to \infty$.*

(b) $(f_\alpha)_{\alpha\in I}$ *has a* ***finite maximal order*** $\kappa_f \geq 2$ ***of topological symmetry****, and for* $\mathbb{P}$*-almost every* $(\alpha_i)_{i\in\mathbb{Z}} \in \Omega$ *there exists*

- *a "globally repelling set"* $R((\alpha_i)_{i\geq 0}) \subset S^1$ *consisting of* κ_f *points, such that for any compact subset* G *of a connected component of* $S^1 \setminus R$*,* $\operatorname{diam}(f_{\alpha_N} \circ \ldots \circ f_{\alpha_0}(G)) \to 0$ *as* $N \to \infty$*;*

  [So if you were to take a high-density ensemble of initial conditions in S^1 and simulate for a long time its evolution under i.i.d. iterates of $(f_\alpha)_{\alpha\in I}$, **you would see** κ_f **clusters of trajectories**; by contrast, in case (a) where there is no topological symmetry, you would only see one cluster.]
- *a "globally backward-time-repelling set"* $A((\alpha_i)_{i<0}) \subset S^1$ *consisting of* κ_f *points, such that for any compact subset* G *of a connected component of* $S^1 \setminus A$*,* $\operatorname{diam}((f_{\alpha_{-1}} \circ \ldots \circ f_{\alpha_{-N}})^{-1}(G)) \to 0$ *as* $N \to \infty$*.*

(c) $(f_\alpha)_{\alpha\in I}$ *is* ***rigid****.*

(B) *Let* $(f_\alpha)_{\alpha\in I}$ *be a* ***non-minimal RCH****. Then there exist* $\underline{\kappa_f \in \mathbb{N} \text{ and } l_f \in \{0,\ldots,\kappa_f - 1\}}$ *such that all of the following holds: There is a mutually disjoint set of* κ_f *non-empty connected closed proper subsets of* S^1*, denoted* $G_0, \ldots, G_{\kappa_f - 1}$ *listed anticlockwise round the circle, such that*

$$f_\alpha(G_i) \;\subset\; G_{i+l_f \bmod \kappa_f}$$

for all i *and* α*; and for* $\mathbb{P}$*-almost every* $(\alpha_i)_{i\in\mathbb{Z}} \in \Omega$ *there exists*

- *a "globally repelling set"* $R((\alpha_i)_{i\geq 0}) \subset S^1$ *consisting of one point in each connected component of* $S^1 \setminus \bigcup_{i=0}^{\kappa_f - 1} G_i$*, such that for any compact subset* G *of a connected component of* $S^1 \setminus R$*,* $\operatorname{diam}(f_{\alpha_N} \circ \ldots \circ f_{\alpha_0}(G)) \to 0$ *as* $N \to \infty$*;*
- *a "globally backward-time-repelling set"* $A((\alpha_i)_{i<0}) \subset S^1$ *consisting of one point in the interior of each of the sets* $G_0, \ldots, G_{\kappa_f - 1}$*, such that for any compact subset* G *of a connected component of* $S^1 \setminus A$*,* $\operatorname{diam}((f_{\alpha_{-1}} \circ \ldots \circ f_{\alpha_{-N}})^{-1}(G)) \to 0$ *as* $N \to \infty$*.*

Definition. From Proposition 1 we define the following numbers associated with a RCH:

(1) For any RCH $(f_\alpha)_{\alpha\in I}$, define the **clustering number** $\kappa_f \in \overline{\mathbb{N}}$ as follows:

- If $(f_\alpha)_{\alpha\in I}$ is minimal and not rigid, then let κ_f be as in Proposition 1(A), with $\kappa_f := 1$ in case (a) where $(f_\alpha)_{\alpha\in I}$ has no topological symmetry.
- Likewise if $(f_\alpha)_{\alpha\in I}$ is non-minimal, then let κ_f be as in Proposition 1(B).
- If $(f_\alpha)_{\alpha\in I}$ is rigid then let $\kappa_f = \infty$.

(2) For a non-minimal RCH $(f_\alpha)_{\alpha\in I}$, define the **rotating number** $l_f \in \{0, \ldots, \kappa_f - 1\}$ as in Proposition 1(B).

(Note that $\kappa_f = \infty$ if and only if $(f_\alpha)_{\alpha\in I}$ is rigid.)

Again, except in the case that (f_α) is rigid, the clustering number κ_f represents the number of clusters you would see if you simulated, over a long time, i.i.d. iterations of the random map (f_α) on a high-density ensemble of initial conditions in S^1.

(The "rotating number" l_f can be linked with the classical notion of *rotation numbers* of orientation-preserving homeomorphisms in the following way: if $(f_\alpha)_{\alpha\in I}$ is non-minimal, then for every $\alpha \in I$ the map f_α has rotation number $\left[\frac{l_f}{\kappa_f}\right]$. But no statement can be made about rotation numbers in the case that $(f_\alpha)_{\alpha\in I}$ is minimal.)

Example 2 ("Prototypical examples")**.** Let $I = [-1, 1]$ with ν the normalised Lebesgue measure. Fix $\varepsilon \in (0, \frac{1}{2\pi})$, and integers $k \in \mathbb{N}$ and $l \in \{0, \ldots, k-1\}$. For all $\lambda \in \mathbb{R} \setminus \{0\}$ define $(f_\alpha^{(\lambda)})_{\alpha \in I}$ by

$$\boxed{f_\alpha^{(\lambda)}([x]) \ = \ \left[x - \tfrac{\varepsilon}{k}\sin(2\pi kx) + \tfrac{l+\lambda\alpha}{k}\right].}$$

- k is the maximal order of rotational symmetry, and also of topological symmetry.
- The "unperturbed map" $f([x]) = \left[x - \frac{\varepsilon}{k}\sin(2\pi kx) + \frac{l}{k}\right]$ has an attractor of k equally spaced points $[0], [\frac{1}{k}], \ldots, [\frac{k-1}{k}]$, and sends each of these points $[\frac{i}{k}]$ to $[\frac{i+l}{k}]$.
- The RCH $f_\alpha^{(\lambda)}$ is formed simply by adding the "noise term" $[\frac{\lambda\alpha}{k}]$ to the unperturbed map.
- The value of $|\lambda|$ represents the **noise intensity**.
- If $|\lambda| > \varepsilon$ then $(f_\alpha^{(\lambda)})_{\alpha \in I}$ is minimal, and $\kappa_{f^{(\lambda)}} = k$.
- If $|\lambda| \leq \varepsilon$ then $(f_\alpha^{(\lambda)})_{\alpha \in I}$ is non-minimal, and $\kappa_{f^{(\lambda)}} = k$ and $l_{f^{(\lambda)}} = l$.

6 Main result

We will

- first state the result for topologically conjugate dynamics (where we are unconcerned about respecting any pre-defined orientation on the topological circle S^1);
- and then state the result for orientationally conjugate dynamics (where we do wish to respect the possibility that S^1 is equipped with a natural choice of orientation).

Note that **if $(f_\alpha)_{\alpha \in I}$ is minimal, then the statement that $\kappa_f = 1$ is equivalent to the statement that $(f_\alpha)_{\alpha \in I}$ has no topological symmetry.**

6.1 Classification up to topologically conjugate dynamics

Theorem 3. *Two RCHs $(f_\alpha)_{\alpha \in I}$ and $(g_\alpha)_{\alpha \in I}$ have topologically conjugate dynamics if and only if at least one of the following three statements holds:*

(i) $(f_\alpha)_{\alpha \in I}$ and $(g_\alpha)_{\alpha \in I}$ are deterministically topologically conjugate [the "trivial" case]*;*

(ii) $\kappa_f = \kappa_g = 1$;

(iii) $(f_\alpha)_{\alpha \in I}$ and $(g_\alpha)_{\alpha \in I}$ are both non-minimal, and $\kappa_f = \kappa_g$ and $l_g \in \{l_f, \kappa_f - l_f\}$.

Note that in case (iii), we don't quite require that $l_f = l_g$, but also allow that l_g is the negative of l_f modulo κ_f. This is due to the fact that topological conjugacy does not have to respect orientation.

Now what's weird about the above theorem is that for RCHs possessing topological symmetry, minimality strangely destroys conjugacy (outside the trivial case of deterministic conjugacy). We give an example to illustrate this (and will give another later).

Example 4. In Example 2, take $l = 0$. For distinct $\lambda_1, \lambda_2 > 0$, we consider the pair of RCHs $(f_\alpha^{(\lambda_1)})_{\alpha \in I}$ and $(f_\alpha^{(\lambda_2)})_{\alpha \in I}$; that is,

$$\begin{aligned} f_\alpha^{(\lambda_1)}([x]) &= \left[x - \tfrac{\varepsilon}{k}\sin(2\pi kx) + \tfrac{\lambda_1\alpha}{k}\right] \\ f_\alpha^{(\lambda_2)}([x]) &= \left[x - \tfrac{\varepsilon}{k}\sin(2\pi kx) + \tfrac{\lambda_2\alpha}{k}\right]. \end{aligned}$$

The unperturbed map is a very "nice" map (it has a simple "global hyperbolic structure", namely that the whole circle is covered by heteroclinic connections between hyperbolic repelling fixed points and hyperbolic attracting fixed points). So one might hope that $(f_\alpha^{(\lambda_1)})_{\alpha\in I}$ and $(f_\alpha^{(\lambda_2)})_{\alpha\in I}$ have topologically conjugate dynamics for small λ_1 and λ_2. Indeed this is so; the full picture is as follows:

(A) In the case that $k = 1$, $(f_\alpha^{(\lambda_1)})_{\alpha\in I}$ and $(f_\alpha^{(\lambda_2)})_{\alpha\in I}$ have topologically conjugate dynamics. (From the point of view of bifurcation analysis: **there are no bifurcations**.)

(B) But in the case that $k \geq 2$ [meaning that we **simply lift the "$k = 1$ case" through order-k rotational symmetry**], $(f_\alpha^{(\lambda_1)})_{\alpha\in I}$ and $(f_\alpha^{(\lambda_2)})_{\alpha\in I}$ have topologically conjugate dynamics if and only if $\lambda_1, \lambda_2 \leq \varepsilon$.

In other words, *conjugacy persists in the $k = 1$ case even as the noise intensity rises above ε, but this persistence beyond ε cannot be lifted to the $k \geq 2$ case.*

6.2 Classification up to orientationally conjugate dynamics

We now move on to the case where conjugacies *do* have to respect the potential choice of a pre-defined orientation of S^1.

Theorem 5. *Two RCHs $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$ have orientationally conjugate dynamics if and only if at least one of the following four statements holds:*

(i) $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$ are deterministically orientationally conjugate [the "trivial" case]*;*

(ii) $\kappa_f = \kappa_g = 1$;

(iii) $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$ are both non-minimal, and $\kappa_f = \kappa_g$ and $l_f = l_g$;

(iv) $(f_\alpha)_{\alpha\in I}$ is minimal, $\kappa_f = 2$, and $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$ are deterministically topologically conjugate.

Recall that for minimal RCHs having topological symmetry, topologically conjugate dynamics is not possible except when the RCHs are deterministically topologically conjugate. Now here, for minimal RCHs having topological symmetry, there is some small hope of orientationally conjugate dynamics without being deterministically orientationally conjugate; but this hope is very peculiar:

Corollary 6. *Suppose $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$ are minimal RCHs each having topological symmetry of order 2 but not higher order; and suppose that $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$ are not deterministically orientationally conjugate. Then $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$ have orientationally conjugate dynamics* ***if and only if*** *there exists an orientation-**reversing** homeomorphism $h\colon S^1 \to S^1$* [intuitively, a re-calibration of S^1 that switches the orientation] *such that for all $\alpha \in I$, $f_\alpha = h^{-1} \circ g_\alpha \circ h$.*

Example 7. In Proposition 2, take $l = 0$. For $\lambda > 0$, we consider the pair of RCHs $(f_\alpha^{(\lambda)})_{\alpha\in I}$ and $(f_\alpha^{(-\lambda)})_{\alpha\in I}$, that is

$$\begin{aligned} f_\alpha^{(\lambda)}([x]) &= \left[x - \tfrac{\varepsilon}{k}\sin(2\pi kx) + \tfrac{\lambda\alpha}{k}\right] \\ f_\alpha^{(-\lambda)}([x]) &= \left[x - \tfrac{\varepsilon}{k}\sin(2\pi kx) - \tfrac{\lambda\alpha}{k}\right]. \end{aligned}$$

So the added noise terms, which are each symmetrically distributed around 0, are simply "reflections" of each other. Again, since the unperturbed map has a simple "global hyperbolic structure", one might hope that $(f_\alpha^{(\lambda)})_{\alpha\in I}$ and $(f_\alpha^{(-\lambda)})_{\alpha\in I}$ have orientationally conjugate dynamics for small λ. Again this is so; the full picture is as follows:

(A) In the case that $\underline{k \in \{1,2\}}$, $(f_\alpha^{(\lambda)})_{\alpha\in I}$ and $(f_\alpha^{(-\lambda)})_{\alpha\in I}$ have orientationally conjugate dynamics.

(B) But in the case that $\underline{k \geq 3}$, $(f_\alpha^{(\lambda)})_{\alpha\in I}$ and $(f_\alpha^{(-\lambda)})_{\alpha\in I}$ have orientationally conjugate dynamics if and only if $\lambda \leq \varepsilon$.

In other words, *orientational conjugacy persists in the $k = 1, 2$ cases even as the noise intensity rises above ε, but this persistence beyond ε cannot be lifted to the $k \geq 3$ case.*

So why, in this example, does orientational conjugacy persist beyond $\lambda = \varepsilon$ in the $k = 2$ case?

If $k = 2$ and $\lambda > \varepsilon$, even though the RCHs $(f_\alpha^{(\lambda)})_{\alpha\in I}$ and $(f_\alpha^{(-\lambda)})_{\alpha\in I}$ are not deterministically orientationally conjugate, nonetheless the deterministic reflection map $x \mapsto -x$ is an orientation-reversing homeomorphism via which $f_\alpha^{(\lambda)}$ is transformed to $f_\alpha^{(-\lambda)}$ for all α, and therefore $(f_\alpha^{(\lambda)})_{\alpha\in I}$ and $(f_\alpha^{(-\lambda)})_{\alpha\in I}$ do have orientationally conjugate dynamics.

6.3 A further consideration of topological conjugacy

There is one further example of the same phenomenon illustrated by Example 4 that we will now present; it ultimately makes exactly the same point as Example 4, but is, I think, interesting to observe. *The example might seem a bit strange if presented immediately, so I will first try to provide some conceptual motivation:*

Let $T^2 := S^1 \times S^1$. Given a map $\Phi \colon T^2 \to T^2$, one can ask whether there is some "nice" (as determined by context) explicit coordinate transformation $h \colon T^2 \to T^2$ via which Φ is conjugate to a map of the form $(x_1, x_2) \mapsto (f(x_1), g(x_2))$ for some $f, g \colon S^1 \to S^1$; we refer to this as the problem of "decoupling" Φ. If Φ can be "decoupled", we can then consider what relationships might exist between the resulting component maps f and g (representing the separate dynamics along the two coordinates) – for example, we can ask is *whether the component maps are topologically conjugate to each other.* (I don't know any situations where people actually do ask questions like this, but it doesn't seem a completely absurd question.)

If the answer is that the component maps *are* topologically conjugate to each other, one could then ask about *whether this scenario will persist under a noisy perturbation of the map Φ*. Now when people formulate "toy models" of physical systems, they are often unconcerned about achieving a highly accurate quantitative description – especially when modelling noise; and so they will instead more prioritise mathematical tractability. So from this point of view, we can imagine that the noise is added in such a manner as not to destroy the decoupling. In other words, assume without loss of generality that Φ is already expressed in its "decoupled form" $\Phi(x_1, x_2) = (f(x_1), g(x_2))$ where f and g are topologically conjugate to each other, and consider a random map $(\Phi_\alpha)_{\alpha\in I}$ of the form $\Phi_\alpha(x_1, x_2) = (f_\alpha(x_1), g_\alpha(x_2))$ where the random maps $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$ on S^1 are noise-perturbations of f and g. *The question is then whether $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$ have topologically conjugate dynamics.* Now a simple kind of noise model is **nondegenerate additive noise**: let $I = [-1, 1] \times [-1, 1]$ with $\nu = \tilde{\nu} \otimes \tilde{\nu}$ for some probability measure $\tilde{\nu}$ on $[-1, 1]$ with full support, and then let

$$\Phi_{(\alpha_1,\alpha_2)}(x_1, x_2) \;=\; (f(x_1) + [\lambda_{11}\alpha_1 + \lambda_{12}\alpha_2]\,,\, g(x_2) + [\lambda_{21}\alpha_1 + \lambda_{22}\alpha_2])$$

for some λ_{ij}-values such that the matrix $\Lambda = (\lambda_{ij})$ is invertible.

For our consideration here, let us just take the very simplest case of the above problem with nondegenerate additive noise: namely, suppose that f and g are already the same map, and that Λ is a multiple of the identity matrix. Taking f as the unperturbed map in Example 2 with $l = 0$, this gives:

Example 8. Let $I = [-1,1] \times [-1,1]$ with $\nu = \tilde{\nu} \otimes \tilde{\nu}$ for some probability measure $\tilde{\nu}$ on $[-1,1]$ with full support, and let

$$\begin{aligned} f_{(\alpha_1,\alpha_2)}([x]) &= \left[x - \tfrac{\varepsilon}{k}\sin(2\pi kx) + \tfrac{\lambda\alpha_1}{k}\right] \\ g_{(\alpha_1,\alpha_2)}([x]) &= \left[x - \tfrac{\varepsilon}{k}\sin(2\pi kx) + \tfrac{\lambda\alpha_2}{k}\right], \end{aligned}$$

where we will take $\lambda > 0$. As in Example 7, these two RCHs have the same probability distribution; but unlike Example 7, the added noise terms are not mirror reflections but are instead statistically independent. We ask whether these two RCHs $(f_{(\alpha_1,\alpha_2)})_{(\alpha_1,\alpha_2)\in I}$ and $(g_{(\alpha_1,\alpha_2)})_{(\alpha_1,\alpha_2)\in I}$ have topologically conjugate dynamics. I think it's not obvious what one "should expect" the answer to this question to be: intuitively, one might imagine that the answer is going to either be 'yes' independently of the values of k, ε and λ, or it will be 'no' independently of the values of k, ε and λ. But what we actually have is:

(A) In the case that $k = 1$, the two RCHs have topologically conjugate dynamics.

(B) But in the case that $k \geq 2$ [meaning again that we simply lift the "$k = 1$ case" through order-k rotational symmetry], the two RCHs have topologically conjugate dynamics if and only if $\lambda \leq \varepsilon$.

So once again, conjugacy exists in the $k = 1$ case regardless of the noise intensity, but for $\lambda > \varepsilon$ the conjugacy cannot be lifted to the $k \geq 2$ case.

Note that the argument for the $k = 2$ case of Example 7 does not apply here, as there is no deterministic topological conjugacy by either an orientation-preserving or an orientation-reversing homeomorphism.

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ADDENDUM: Some further thoughts (after discussion with Jan Sieber)

There are two aspects of the results I presented, that I regarded as surprising and even as revealing that the definition of conjugacy of random dynamical systems does not, in general, match what one would expect for a notion of "conjugacy" of dynamical systems. Namely:

1. It seemed to me strange that in examples like Example 4 and Example 8, the existence of conjugacy for noise intensities larger than ε cannot be lifted from the $k = 1$ case to the case of higher k. To expound further:

   The $k \geq 2$ case simply consists of "fitting into the circle k identical copies of the dynamics of the $k = 1$ case". Or to put it another way, from a more "physical" perspective: As I see it, the stochastic difference equation

$$x_{n+1} = f^{(\lambda)}_{\alpha_n}(x_n) \qquad \{\alpha_n\}_{n\in\mathbb{Z}} \text{ i.i.d.} \sim \text{Uniform}(-1,1)$$

with $k \geq 2$ simply models with a "k-fold redundancy of information" the same process as is modelled "non-redundantly" by the stochastic difference equation

$$x_{n+1} = \tilde{f}^{(\lambda)}_{\alpha_n}(x_n) \qquad \{\alpha_n\}_{n\in\mathbb{Z}} \text{ i.i.d.} \sim \text{Uniform}(-1,1)$$

where $(\tilde{f}^{(\lambda)}_{\alpha})_{\alpha\in I}$ is the $k=1$ version. Therefore, it seemed strange to me that on the one hand, the latter model exhibits no bifurcations as the parameter λ is varied throughout $(0,\infty)$, while on the other hand, the former model does undergo bifurcations as λ rises past a certain critical value.

2. Corollary 6, and its illustration through Example 7 seem strange to me: the idea of a fairly simple situation in which the existence of a deterministic global orientation-*reversing* conjugacy is both the sufficient and the necessary condition for orientation-preserving conjugacy of dynamics seems weird.

Regarding the first point

Considering, as our prototype, Example 2 with $l = 0$ and $\lambda > 0$:

I had been under the impression that any meaningful "change that takes place as λ rises above ε" that does exist in the $k \geq 2$ case while not existing in the $k = 1$ case *is not a change visible on any surface-level consideration of the random maps, but rather only exists in the deeper world of the random dynamical systems formalism (in which a central role is played by the dynamics of the shift map on the bi-infinite product space of the noise).*

However, I had a discussion with Jan Sieber immediately after the last of the three seminars, in which he observed that **this is not the case**. There is an important change for $k \geq 2$, which we can understand in terms of the following equivalence relation (which I will define for very general random maps, although it probably simplifies a little in our case):

Definition. Fix $(I,\mathcal{I},\nu)$ and let $(f_\alpha)_{\alpha\in I}$ be a random map on a compact metric space X. We say that two points $x, y \in X$ are *equivalent* if there exists $z \in X$ such that for every neighbourhood U of z there exist $m, n \geq 0$ such that

$$\nu^{\otimes m}((\alpha_0,\ldots,\alpha_{m-1}) \in I^m : f_{\alpha_{m-1}} \circ \ldots \circ f_{\alpha_0}(x) \in U) > 0$$
$$\nu^{\otimes n}((\alpha_0,\ldots,\alpha_{n-1}) \in I^n : f_{\alpha_{n-1}} \circ \ldots \circ f_{\alpha_0}(y) \in U) > 0.$$

Applying this to our example:

- in the case that $k = 1$, the whole circle is one equivalence class for all λ;
- but in the case that $k \geq 2$, the whole circle is one equivalence class if and only if $\lambda > \varepsilon$.

Hence the results about conjugacy in Example 4 do not indicate as much of a "defficiency from a practical point of view" (if any at all!) in the definition of conjugacy for random dynamical systems as I had previously thought!

Now if we take $l \neq 0$, then our equivalence relation would have to be modified accordingly to illustrate the point, but the basic point stands nonetheless.

Regarding the second point

When I described that the results for orientationally conjugate dynamics in relation to minimal RCHs with $\kappa_f = \kappa_g = 2$ seem strange, Jan Sieber pointed out that two points are not enough to be able to distinguish orientations while three or more points are. This observation is indeed part of what is central to the proof of the result: conjugacies defined purely between the dynamics on the random two-point attractors are automatically both orientation-preserving and orientation-reversing. (Conjugacies on the random attractors play a central role in addressing the existence of global conjugacies.)

Still, it did nonetheless seem strange to me that the way to achieve orientationally conjugate dynamics outside of deterministic orientational conjugacy is precisely the existence of a *global deterministic orientation-reversing conjugacy* – which, intuitively speaking (to my thinking), *does* concern more than just a (stochastic) pair of points, it concerns simultaneous consideration of all of the uncountably infinitely many deterministic points on the circle.

But I did then think about it further: For RCHs with any given clustering number $\kappa \geq 1$, if dynamics on the random κ-point attractors can be (non-deterministically) conjugated via an orientation-preserving or orientation-reversing conjugacy, then it would not be surprising if a consequence is that the conjugacy can be extended to a global (non-deterministic) conjugacy that is respectively orientation-preserving or orientation-revsersing. (Indeed, I'm quite certain that this is the case, and that it follows immediately from parts of the proof of our results.) Hence it would follow that for RCHs with a clustering number of 2 (regardless of minimality), the existence of non-deterministic orientation-preserving conjugacy is equivalent to the existence of non-deterministic orientation-reversing conjugacy. Again, this would not be surprising, and I'm pretty certain that it is true.[2] *Combining this statement with Theorem 3* (the purely topological consideration of conjugacy) *immediately implies the strange-seeming Corollary 6.* **However**, contrary to the order in which I chose to present the results, we did *not* first prove Theorem 3 and then proceed from there to obtain Theorem 5 and Corollary 6. Rather, it was necessary for us first to prove Theorem 5, *a part of which* was to prove (*not* as a corollary of some other theorem) the statement that I have chosen to call "Corollary 6"; and then we obtained Theorem 3 as an immediate consequence of Theorem 5. In other words: what seems (at least to me) the "natural" order of didactic exposition of our results (as in these seminars) is the reverse of the order in which we were able to prove the results. Now the proofs of the two directions of implication in Corollary 6 felt to me "very independent of each other", such that it felt rather "fluky" that this two-way implication, and hence the statement of Theorem 5 (and hence also our proof of Theorem 3), holds. And yet, as I have said, the statement of Corollary 6 would feel like a very natural consequence Theorem 3, if we think the other way round.

These considerations seem to me very much to support something I have already been suspecting, namely that there exists some generalisation (probably a higher-dimensional generalisation) of Theorem 3, for which the proof is probably longer than our proof of Theorem 3 but is nonetheless "more clear" in revealing what is really going on.

[2] Note however that the same statement with deterministic conjugacies in place of non-deterministic conjugacies would be completely false, although the analogous statement for classical conjugacy of maps like the unperturbed map in Example 2 would be true.