- (X, T, μ)
  μ prob. (ergodic)
- $A_n \subset X$.
  Often A_n are shrinking balls.
  A_n called shrinking targets.

Question
When do we have

$$T^n x \in A_n$$

for inf. many n?

Borel - Cantelli

E_n sets. $E = \limsup E_n$

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k .$$

1) $\sum \mu(E_n) < \infty$
$\Longrightarrow \mu(E) = 0$

2) $$\left.\begin{array}{l} \sum \mu(E_n) = \infty \\ + \text{independence} \end{array}\right\} \Rightarrow \mu(E) = 1$$
$(\mu(E_n \cap E_m) = \mu(E_n)\mu(E_m))$

Erdős – Rényi 1959

Replace indep by

$$\liminf_{n\to\infty} \frac{\sum_{j,k=1}^{n}\left(\mu(E_j \cap E_k) - \mu(E_j)\mu(E_k)\right)}{\left(\sum_{k=1}^{n}\mu(E_k)\right)^2}$$

≤ 0.

In dynamics:

$E_n = T^{-n} A_n$

$x \in E_n \Leftrightarrow T^n x \in A_n$

Dynamical Borel – Cantelli.

Allways:

$\sum \mu(A_n) = \sum \mu(E_n) < \infty$

$\Longrightarrow \mu(E) = 0$

For a.e. x we have $T^n x \in A_n$ for at most finitely many n.

Sometimes:

$$\left.\begin{array}{l} \sum \mu(A_n) = \infty \\ + \text{assumptions} \end{array}\right\} \Rightarrow \mu(E) = 1.$$

Let

$$S_n(x) = \sum_{k=1}^{n} \mathbb{1}_{A_k}(T^k x)$$

Then

$$\mathbb{E} S_n = \sum_{k=1}^{n} \mu(A_k)$$

Strong B-C.:

$$\left.\begin{array}{l} \text{assumptions} \\ \sum \mu(A_n) = \infty \end{array}\right\} \Rightarrow \lim_{n\to\infty} \frac{S_n(x)}{\mathbb{E} S_n} = 1$$

for a.e. x.

Fractal stuff

Assume $\sum \mu(A_n) < \infty$.

Hausdorff dimension of

$$\{x : T^n x \in A_n \text{ for } \infty \text{ many } n\}?$$

Assume that (X, T, μ) has exp. decay of correlations for L^1 against BV:

$$\left| \int f \circ T^n g \, d\mu - \int f \, d\mu \int g \, d\mu \right|$$

$$\leq C \|f\|_1 \|g\|_{BV} \vartheta^n$$

μ $\qquad \vartheta \in (0,1)$.

$\|g\|_{BV} = \sup|g| + \operatorname{var} g$

for instance if

T is p.w. exp interval map.

$$T(x) = \frac{1}{x} \mod 1$$

Theorem If A_n are intervals then

$$\sum \mu(A_n) = \infty \implies$$

$\Rightarrow$ for a.e. x

$T^n x \in A_n$ for ∞ many n.

(actually also strong B–C.)

Proof $\quad S_n = \sum_{k=1}^{n} \mathbb{1}_{A_k} \circ T^k$

Let

$$Y_n = \frac{S_n}{\mathbb{E} S_n} - 1 = \frac{S_n - \mathbb{E} S_n}{\mathbb{E} S_n}$$

Then

$$\mathbb{E}\left(Y_n^2\right) = \frac{1}{(\mathbb{E} S_n)^2}\left(\underline{\mathbb{E}(S_n^2) - (\mathbb{E} S_n)^2}\right)$$

$$\mathbb{E}(S_n^2) - (\mathbb{E} S_n)^2 =$$

$$= \sum_{k,j=1}^{n}\left(\mathbb{E}\left(\mathbb{1}_{A_k} \circ T^k\, \mathbb{1}_{A_j} \circ T^j\right) - \mu(A_k) \times \mu(A_l)\right)$$

$$= \sum_{k=j} + 2 \sum_{j<k}$$

$$\leq \sum_{k=1}^{n} \mu(A_k) + 2 \sum_{j<k} \left(\int \mathbb{1}_{A_k} \circ T^{k-j}\, \mathbb{1}_{A_j}\, d\mu - \int \mathbb{1}_{A_k} d\mu \int \mathbb{1}_{A_j} \right)$$

$$\leq \mathbb{E} S_n + 2 \sum_{j<k} C \theta^{k-j} \mu(A_k) 3$$

$$\leq \mathbb{E} S_n + C' \, \mathbb{E} S_n$$

$$= \underline{(1 + C')\, \mathbb{E} S_n}\,.$$

$$\Longrightarrow \mathbb{E}\left(Y_n^2\right) \leq \frac{1 + C'}{\mathbb{E} S_n}$$

$$\longrightarrow 0 \text{ as } n \to \infty.$$

Take a sequence n_k s.t.

$$\sum_{k=1}^{\infty} \mathbb{E}(Y_{n_k}^2) < \infty$$

$\Rightarrow$ For a.e. x

$$Y_{n_k}^2(x) \to 0$$

$\Rightarrow$ For a.e. x

$$Y_{n_k}(x) \to 0.$$

$\Rightarrow$ For a.e. x

$$S_{n_k}(x) \to \infty$$

$$S_n(x) \to \infty.$$

If $\boxed{n_k = k}$ then

$$\frac{S_n(x)}{\mathbb{E} S_n} \to 1 \text{ for a.e. } x.$$

Lemma (Schmidt–Sprindžuk)

f_k are functions ≥ 0.

a_k are numbers.

φ_k ______

If for all m, n:

$$\int \left(\sum_{k=m}^{n} (f_k - a_k) \right)^2 d\mu \leq \sum_{k=m}^{n} \varphi_k$$

then for almost all x:

$$\sum_{k=1}^{n} f_k(x) = \sum_{k=1}^{n} a_k + \text{remainder}$$

where
remainder =
$$O\left(\Phi(n)^{\frac{1}{2}}(\log \Phi(n))^{3/2+\varepsilon}\right)$$
$$\Phi(n) = \sum_{k=1}^{n} \varphi_k .$$

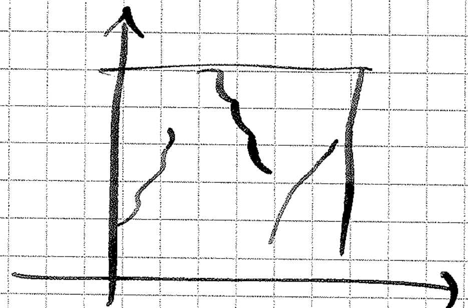
Here :
$$f_k = \mathbb{1}_{A_k} \circ T^k$$
$$a_k = \mathbb{E} f_k = \mu(A_k)$$
$$\varphi_k = a_k \, (?)$$

You get :

$$S_n(x) \sim \mathbb{E} S_n + O\left(\sqrt{\mathbb{E} S_n} \log^{3/2}\right)$$

(Kim 2007).

Gauß-map.
p.w. $2x \bmod 1$
$\beta x \bmod 1$



References

- Philipp 1967.

  $T(x) = \frac{1}{x} \mod 1$

  $T(x) = \beta x \mod 1 \quad \beta > 1$

  Strong B–C: A_n interv.

  $$\frac{\sum_{k=1}^{n} \mathbb{1}_{A_k}(T^k x)}{\sum_{k=1}^{n} \mu(A_k)} \to 1 \text{ a.e.}$$

- Dolgopyat 2004.

  Strong BC

  $T =$ Anosov diff

  $\mu \ll$ Leb.

- Chernov – Kleinbock 2001.

  Anosov diff.

  $A_n =$ cylinder sets.

  Also proved in general:

  (X, T, μ)

  There is always A_n s.t.

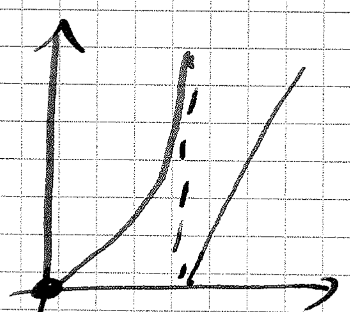
  $$\sum \mu(A_n) = \infty$$

  but for a.e. x

  $T^n x \in A_n$ for only finitely many n.

- Kim 2007 $\alpha > 0$

$$T_\alpha(x) = \begin{cases} x(1+2^\alpha x^\alpha) & [0,\tfrac{1}{2}) \\ 2x-1 & [\tfrac{1}{2},1) \end{cases}$$



Slowly mixing $0<\alpha<1$.

$\exists$ intervals A_n s.t

$\sum \mu(A_n) = \infty$

but $T^n x \in A_n$ for finitely many n.

- Gouëzel 2007
Same system
$\sum \cancel{\mu}\ Leb(A_n) = \infty$
$\Rightarrow$ BC.

- Gupta, Nicol, Ott 2010.
Strong Borel–Cantelli for 1D Gibbs–Markov.

- Haydn – Nicol – Persson – Vaienti 2013.
BC and strong BC
Systems with decay of corr. for Hölder cts functions.

$\mathbb{1}_{A_k} \approx$ Hölder cts.
$\mu(A_n) \geq \frac{1}{n^\gamma}$

- Xing 2021
Improved last two papers,

$A_n = B(y, r_n)$

$T^n x \in A_n \iff$

$d(T^n x, y) < r_n$.

what about

$d(T^n x, x) < r_n$?

Recurrence. (X, T, μ)

Poincaré' ~~rec~~ recurrence theorem (Caratheodory 1919)

If X is a separable metric space, then

$\liminf_{n\to\infty} d(T^n x, x) = 0$.

for μ a.e. x.

Boshernitzan 1993

If the α-dim Hausdorff measure on X is σ-finite then

$\liminf_{n\to\infty} n^{1/\alpha} d(T^n x, x) < \infty$

for μ a.e. x

(means that for μ a.e. x

$d(T^n x, x) < \frac{c}{n^{1/\alpha}}$ for

∞ many n.)

$\left(\mu(B_r) \sim r^\alpha \right.$

$\left. \mu\left(B\left(x, \frac{c}{n^{1/\alpha}}\right)\right) \sim \frac{c}{n} \right)$

If α-dim Hausd. m. is finite then

$\liminf \ldots = 0$.

- Pawelec 2017

  Expanding maps μ s.t. α-dim Packing measure is σ-finite on a set A, with $\mu(A) = 1$ + exp. mixing.

  Then

  $$\liminf_{n\to\infty} (n \log\log n)^{1/\alpha} d(T^n x, x) = 0.$$

  for μ a.e. x.

- Chang – Wu – Wu 2019
- Baker – Farmer 2021

  X = attr. of self sim set μ – natural

  Then

  $$\left.\begin{array}{l} T^n x \in B(x, r_n) \\ \text{inf. many } n \\ \text{and a.e. } x \end{array}\right\} \Leftrightarrow \sum r_n^\alpha = \infty$$

  $\alpha = \dim X$.

Kirsebom – Kunde – Persson

T = exp. linear map of $\mathbb{T}^d$.

μ = Leb.

$$\left.\begin{array}{l} T^n x \in B(x, r_n) \\ \text{for inf. many } n \\ \text{a.e. } x \end{array}\right\} \Leftrightarrow \sum r_n^d = \infty$$

Hussein – Li – Simmons – Wang

P.W. expanding and conformal system in particular for Gauß-map

Let

$$E_n = \{x : d(T^n x, x) < r_n\}.$$

One can estimate $\mu(E_n)$.

$$\mu(\limsup_{n\to\infty} E_n) = ?$$

$$\mu(E_j \cap E_k) \overset{?}{\leq} H \, \mu(E_j)\mu(E_k) + \text{small}, \quad H > 1$$

E_j has more compl. structure

$$\mu(\limsup) \geq \frac{1}{H}$$