Attractors and attracting measures

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TIPPING POINTS IN THE EARTH SYSTEM

Purpose

The purpose of this talk is not just to explain some of what I've been thinking about with Pete, but also potentially to get some feedback and learn.

$\rightarrow$ I have pretty much no experience with smooth ergodic theory.

I will probably be "asking" at least as many questions as I "answer".

Motivation

Deterministic model of a climate system “without climate change”:

- autonomous DE $\dot{x} = F(x)$ on some state space M, ...
  - $\rightarrow$ autonomous since climate parameters are not changing;
  - $\rightarrow$ but there may be different regions of M corresponding to “qualitatively different” climate scenarios, so:
- ...with some given “attractor” $A \subset M$ representing a stable qualitative state of the climate system.

E.g. AMOC could be described by an ODE $\dot{x} = F(x)$ with an attractor A_{on} corresponding to the “on” state of the AMOC (while there is another attractor A_{off} corresponding to a qualitatively different stable state of the AMOC, the “off” state).

Motivation

Without "climate change", the quantitative state of the climate is always changing!

$\Rightarrow$ the attractor A is not a single point.

Leads to the question:

> *"If I observe this deterministic climate system at some 'random' time t that has nothing to do with the state of the climate itself, what is the probability distribution for the quantitative climate state $x(t)$ that I will observe?"*

Note:

- This would be a probability distribution supported on A.
- If A is a "chaotic attractor", one would expect the answer to be unaffected by previous observations made sufficiently long ago.

Motivation

Problems:

- When is there a well-defined answer μ to this question?
- In what ways can μ be numerically simulated?
- Introduce climate change:

$$\dot{x}(t) = F_t(x(t)), \qquad \lim_{t \to -\infty} F_t = F$$

→ evolve A forward from time $-\infty$ to get a set-valued trajectory $A(t)$ equipped at each time t with the corresponding probability distribution $\mu(t)$;

→ but this doesn't mean anything—how do we give this rigorous meaning and simulate it?

→ if this system exhibits *"partial tipping"* then we use $\mu(t)$ to define the probability of tipping [Ashwin & N., 2021].

Although the ultimate goal is this "climate-changing" case, we're still trying to understand aspects of the autonomous case.

Axiom A attractors

M – compact Riemannian manifold
m – Lebesgue measure
$(f^t)_{t\in\mathbb{R}}$ – solution flow for an autonomous ODE on M

An Axiom A attractor $A \subset M$ is a “chaotic attractor with very nice hyperbolicity properties”.

[Bowen & Ruelle, 1975][1] Under weak conditions, given an Axiom A attractor A with “basin of attraction”

$$B_A := \{x_0 \in M : d(f^t x_0, A) \to 0 \text{ as } t \to \infty\},$$

A is the support of an <u>ergodic invariant probability measure μ_A</u> with the following two properties:

[1] Analogous result for discrete-time maps in [Ruelle, 1976].

Axiom A attractors

1. [BR75, Thm. 5.1] For m-a.e. $x_0 \in B_A$,

$$\frac{1}{t}\int_0^t \delta_{f^s x_0}\, ds \overset{\text{weakly}}{\longrightarrow} \mu_A \text{ as } t \to \infty.$$

$\to$ LHS $=$ law of $f^T x_0$ for $T \sim \mathrm{Unif}(0,t)$.

Property 1 is analogous to ergodicity: a prob. meas. μ is ergodic iff for μ-a.e. $x_0 \in M$,

$$\frac{1}{t}\int_0^t \delta_{f^s x_0}\, ds \overset{\text{weakly}}{\longrightarrow} \mu.$$

Axiom A attractors

2. [BR75, Thm. 5.3] For every p.m. $\nu_0 \ll m$ with $\nu_0(B_A) = 1$,

$$f^t \nu_0 \overset{\text{weakly}}{\to} \mu_A \text{ as } t \to \infty.$$

$\to$ LHS $=$ law of $f^t X_0$ for $X_0 \sim \nu_0$.

Property 2 is analogous to mixing: a p.m. μ is mixing iff $\forall$ p.m. $\nu_0 \ll \mu$,

$$f^t \nu_0 \underset{\text{(or strongly)}}{\overset{\text{weakly}}{\to}} \mu.$$

(**Remark.** mixing $\Rightarrow$ ergodic.)

Axiom A attractors

The analogy between Property 2 and mixing can be understood in terms of decay of “classical correlations” vs. decay of “operational correlations” (e.g. [Baladi *et al.*, 2002]):

- An invariant p.m. μ is mixing iff $\forall\, g_1, g_2 \in C_b(M, \mathbb{R})$,

$$\mathrm{Cov}_\mu[g_1, g_2 \circ f^t] = \int g_1(x) g_2(f^t x)\, \mu(dx) - \int g_1\, d\mu \int g_2\, d\mu \;\to\; 0.$$

- Property 2 can be re-expressed as: $\forall\, g_1, g_2 \in C_b(B_A, \mathbb{R})$,

$$\mathrm{OC}(g_1, g_2, t) := \int_{B_A} g_1(x) g_2(f^t x)\, m(dx) - \int_{B_A} g_1\, dm \int g_2\, d\mu_A \;\to\; 0.$$

Axiom A attractors

Note: in [BR75], μ_A is constructed in Sec. 3; then Thms. 5.1 and 5.3 are separately proved based on material developed prior to Sec. 5.

Remark:

- Pr. 1 holds in the general setting of [BR75].
- Pr. 2 holds under mild extra assumption: the unstable manifold of each point in A is dense in A.

$\rightarrow$ Extra assumption only enters via the fact that it implies μ_A is mixing.

So [BR75, Thm. 5.3] “really” says: μ_A mixing $\Rightarrow$ μ_A has Pr. 2.

“Attracting measures”

Property 1: “physical measures”

$\rightarrow$ extensively studied in more general settings

Property 2: ???

$\rightarrow$ I will call such measures attracting measures

"Attracting measures"

When A is a Lebesgue null set:

- mixing—purely in and of itself—may not be of much physical relevance;
- "attracting" is probably the more physically accessible notion of mixing dynamics [Baladi *et al.*, 2002];
- and yet it seems that attracting measures are under-appreciated and/or under-studied!

E.g. the Lorenz system

- has a chaotic attractor A supporting a physical measure μ_A [Tucker, 2002];
- μ_A is mixing at an exponential rate [Araújo & Melbourne, 2016];
- but is it known whether μ_A is attracting??

What I will do in this talk

I will

- present definitions of attractors, physical measures and attracting measures in a generalised setting;
- present a generalisation of [BR75, Thm. 5.3];
- raise several questions along the way and at the end.

My generalised setting

M – Riemannian manifold
m – Lebesgue measure
d – geodesic distance
$(f^t)_{t \geq 0}$ – continuous semiflow of C^1 local diffeomorphisms

Actually, I can make it even more general (purely topological):

M – Polish space
m – locally finite measure of full support
d – metrisation of the topology of M
$(f^t)_{t \geq 0}$ – continuous semiflow of open mappings $f^t \colon M \to M$ admitting a "well-defined transfer operator that locally respects boundedness" (every $x \in M$ has a nbhd U s.t. for all $t \geq 0$, $m(U \cap f^{-t}(\cdot))$ is m-abs. cont. with bounded density).

Physical and attracting measures

A p.m. μ whose support A is compact is called a(n)

- physical measure if $\exists$ nbhd $U \supset A$ s.t. for m-a.e. $x_0 \in U$,

$$\frac{1}{t}\int_0^t \delta_{f^s x_0}\, ds \overset{\text{weakly}}{\longrightarrow} \mu;$$

- attracting measure if $\exists$ nbhd $U \supset A$ s.t. for each p.m. $\nu_0 \ll m$ with $\nu_0(U) = 1$,

$$f^t \nu_0 \overset{\text{weakly}}{\longrightarrow} \mu.$$

I will generalise the proof of [BR75, Thm. 5.3] to obtain general conditions under which mixing implies attracting.

Attractors

A compact set $A \subset M$ with $f^t A = A$ for all $t \geq 0$ is called a

A. pointwise attractor if $\exists$ nbhd U s.t.

$$d(f^t x_0, A) \to 0 \text{ for each } x_0 \in U;$$

B. uniform attractor if $\exists$ nbhd U s.t.

$$d(f^t x_0, A) \to 0 \text{ uniformly across } x_0 \in U;$$

C. pointwise attractor via stable manifolds if $\exists$ nbhd U and $\pi : U \to A$ [w.l.o.g. Lebesgue-measurable] s.t.

$$d(f^t x_0, f^t \pi(x_0)) \to 0 \text{ for each } x_0 \in U;$$

Attractors

- **D** uniform attractor via stable manifolds if $\exists$ nbhd U and $\pi : U \to A$ s.t.

$$d(\, f^t x_0 \,,\, f^t \pi(x_0)\,) \to 0 \text{ uniformly across } x_0 \in U;$$

$\to$ meaning: for any $\varepsilon > 0$, taking sufficiently large t gives

$$\sup_{x_0 \in U} d(\, f^t x_0 \,,\, f^t \pi(x_0)\,) < \varepsilon.$$

Weaker version: let π depend on ε, namely

- **E** uniform attractor via shadowing if $\exists$ nbhd U s.t. for any $\varepsilon > 0$, $\exists\, \pi_\varepsilon : U \to A$ s.t. taking sufficiently large t gives

$$\sup_{x_0 \in U} d(\, f^t x_0 \,,\, f^t \pi_\varepsilon(x_0)\,) < \varepsilon.$$

Axiom A case

An Axiom A attractor is both pointwise via stable manifolds and uniform via shadowing [BR75, Prop. 4.4].

Questions:

1. What about uniform via stable manifolds?
2. Can π (in def'n of pointwise via stable manifolds) be chosen to be continuous?
3. Can (U, π) be chosen s.t. $\pi(m|_U) \ll \mu_A$?

A trivial result

Proposition

Suppose we have p.m. μ *whose support A is a pointwise attractor via stable manifolds.*

Suppose (U, π) *can be chosen s.t.* $\pi(m|_U) \ll \mu$.

If μ *is mixing then* μ *is attracting.*

Using our further-above characterisation of mixing, the proof is a trivial application of the dominated convergence theorem.

But I suspect that this result is useless (i.e. conditions typically don't hold or are very difficult to verify)??

A maybe more useful result

Theorem (generalising [BR75, Thm. 5.3])

Suppose we have p.m. μ whose support A is a uniform attractor via shadowing.

Suppose μ is mixing. Suppose μ also satisfies $(*)$:
$\exists$ *arbit'ly small $\varepsilon > 0$ s.t. one can find an unbounded set $\mathcal{T}_\varepsilon \subset [0,\infty)$ with*

$$\inf_{T\in\mathcal{T}_\varepsilon, x\in A} \frac{\mu(y \in A \,:\, d(f^t x, f^t y) < \varepsilon \;\; \forall\, t \in [0,T])}{m(y \in M \,:\, d(f^t x, f^t y) < 3\varepsilon \;\; \forall\, t \in [0,T])} > 0.$$

Then μ is attracting.

[BR75] uses 2ε in place of 3ε (and verifies $(*)$ in Cor. 4.6 with $\mathcal{T} = [0,\infty)$ for all suff. small ε), but the proof doesn't seem to work with 2ε.

Some further questions

- Can we find settings outside of the Axiom A setting in which the Theorem can be applied?
- Can the conditions of the Theorem be weakened/modified so as to be more easily applicable beyond the Axiom A setting?
- In particular, might it be the case generally that every mixing physical measure is attracting?

Some further questions

- It is known that if a probability measure μ is an SRB measure in the sense of [Young, 2002], then it is “physical” under a weaker definition where U may now be any m-positive-measure set [Pugh & Shub, 1989].

$\hookrightarrow$ If a probability measure μ is an SRB measure in the sense of [Young, 2002] and is also mixing, does it follow that μ is an “attracting measure”, at least under a weaker definition where U may now be any m-positive-measure set?

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Thank you.