Path-dependent, shrinking, moving targets and beyond, on generic self-affine sets

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The game

The game

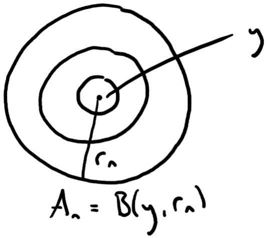
- $A_n \subset X$ and (T, X),

$$\{x \in X \mid T^n(x) \in A_n \text{ for infinitely many } n\}$$

Question

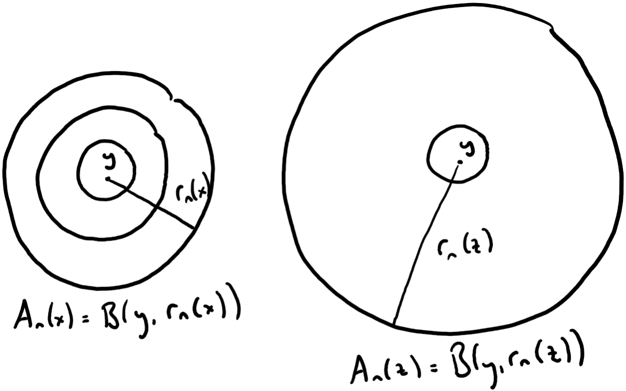
How big is the shrinking target set for a given sequence A_n?

Shrinking targets



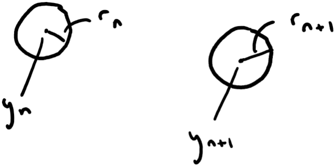
- $\limsup\{x \mid T^n(x) \in A_n\}$

Path-dependent target size



- $\limsup\{x \mid T^n(x) \in A_n(x)\}$

Moving targets



- $A_n = B(y_n, r_n)$

Some earlier work

- Hill and Velani
- Bugeaud and Wang
- Li, Wang, Wu, Xu
- Reeve
- K. and Ramìrez
- Barany and Rams

Space and time

Self-affine sets

- iterated function system: $f_1, \ldots, f_N$ affine contractions $f_i = A_i + a_i$
- self-affine set: unique, non-empty, compact

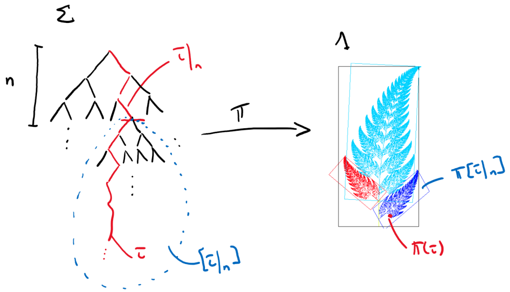
$$\Lambda = \bigcup_{i=1}^{N} f_i(\Lambda)$$

- assumption: the union is disjoint (strong separation condition) and $|A_i| < \frac{1}{2}$

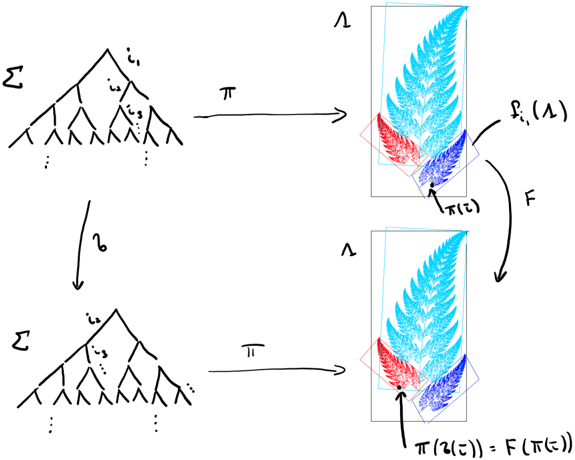
Symbolic dynamics

- symbolic space: $\Sigma = \{1, \ldots, N\}^{\mathbb{N}}$
- shift: $\sigma(i_1, i_2, i_3, \ldots) = (i_2, i_3, \ldots)$
- projection: $\pi : \Sigma \to \Lambda$ (bijection)
- cylinder:

$$[\mathbf{i}|_n] = \{\mathbf{j} \in \Sigma \mid \mathbf{j}|_n = \mathbf{i}|_n\}$$



Expanding dynamics



- $T : \Lambda \to \Lambda : \pi \circ \sigma = T \circ \pi$

Dimensions of self-affine sets

- Hausdorff measure

$$\mathcal{H}^s(\Lambda) = \lim_{\delta \to 0} \inf\{\sum_{i=1}^{\infty} \mathrm{diam}(B_i)^s \mid \Lambda \subset \cup_{i=1}^{\infty} B_i, \mathrm{diam} B_i < \delta\}$$

- Hausdorff dimension

$$\dim_H \Lambda = \inf\{s \mid \mathcal{H}^s(\Lambda) = 0\}.$$

Dimensions of self-affine sets

- length of a finite word $\mathbf{i}$: $|\mathbf{i}|$
- pressure

$$P(s) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\mathbf{i}|=n} \phi^s(\mathbf{i}),$$

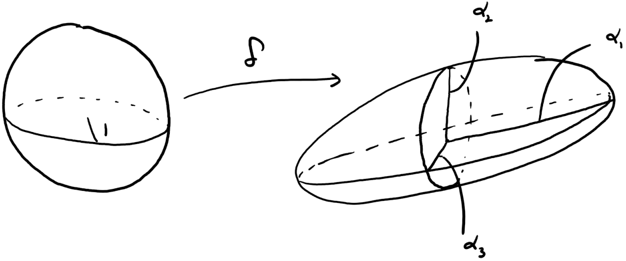
- $\phi^s(\mathbf{i})$ the singular value function of $f_{\mathbf{i}} = f_{i_1} \cdots f_{i_n}$

Theorem (Falconer, Solomyak)

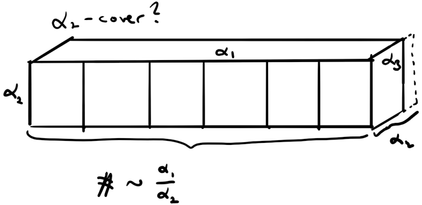
Assume $|A_i| < \frac{1}{2}$. For Lebesgue almost all $a_1, \ldots, a_N$

$$\dim_H \Lambda = s_0 : P(s_0) = 0$$

Upper bound for dimension: singular value function



- $\phi^s(f) = \alpha_1 \dots \alpha_{\lceil s \rceil} \alpha_{\lfloor s \rfloor}^{\{s\}} \sim \# \cdot \alpha_{\lfloor s \rfloor}^s \Rightarrow \mathcal{H}^s$



Shrinking targets on self-affine sets

- Recall $T : \Lambda \to \Lambda$, $T \circ \pi = \pi \circ \sigma$
- symbolic shrinking target set: $\mathbf{j} \in \Sigma$, $\ell_n(\mathbf{i}) \to \infty$

$$R(\mathbf{j}) = \{\mathbf{i} \in \Sigma \mid \sigma^n(\mathbf{i}) \in [\mathbf{j}|_{\ell_n(\mathbf{i})}] \text{ for infinitely many } n\}$$

- shrinking target set on Λ: $\pi(\mathbf{j}) = y \in \Lambda$,

$$R^*(y) = \{x \in \Lambda \mid T^n(x) \in \pi[\mathbf{j}|_{\ell_n(\mathbf{i})}] \text{ for infinitely many } n\}$$

- satisfies:

$$R^*(y) = \pi(R(\mathbf{j}))$$

Michal, 09:43

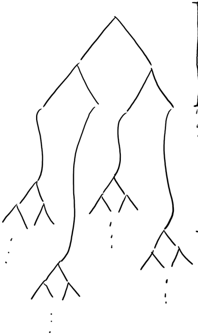
well, I know things are varying there over time and space, but I do not grasp the flow

Michal is typing

The story

Lower bounds for shrinking target sets

- $$R(\mathbf{j}) = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} \{\mathbf{i} \in \Sigma \mid \sigma^n(\mathbf{i}) \in [\mathbf{j}|_{\ell_n(\mathbf{i})}]\}$$
- dimension lower bounds via
  - Cantor subset $C \subset R(\mathbf{j})$
  - mass distribution μ on C with $\pi_*\mu$ of finite energy



K. and Ramìrez

Recall Falconer: for Lebesgue almost all translations, the dimension of a self-affine Λ is

$$s_0 : P(s_0) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\mathbf{i}|=n} \phi^{s_0}(\mathbf{i}) = 0$$

- **non-path-dependent target size:** $\ell_n(\mathbf{i}) = \ell_n \to \infty$
- modified pressure:

$$P(s, \mathbf{j}) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\mathbf{i}|=n} \phi^{s}(\mathbf{ij}|_{\ell_n})$$

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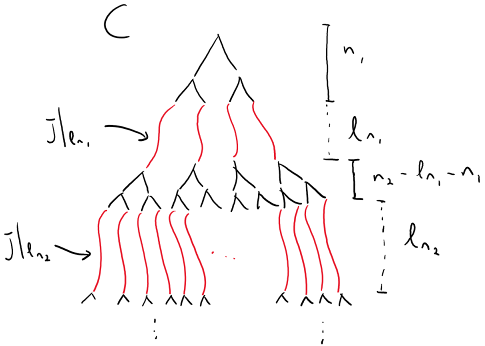
Theorem (K. and Ramìrez)

Assume A_i satisfy a quasimultiplicativity assumption and $|A_i| < \frac{1}{2}$. For almost all translations with strong separation condition, for typical $\mathbf{j} \in \Sigma$,

$$\dim_H R^*(\pi(\mathbf{j})) = s_0 : P(s_0, \mathbf{j}) = 0.$$

K. and Ramìrez: Cantor set and mass distribution

the mass is distributed according to ϕ^s and



Path-dependent: Friday Theorem

- **path-dependent target size:** $\ell_n(\mathbf{i})$ only depends on the first n symbols and is approximately additive on finite words:

$$|\ell_{|\mathbf{i}|+|\mathbf{k}|}(\mathbf{ik}) - \ell_{|\mathbf{i}|}(\mathbf{i}) - \ell_{|\mathbf{k}|}(\mathbf{k})| < K$$

- $\mathbf{j}$ a typical point for some ergodic measure on Σ: $Z(\mathbf{j}) = \lim_{n\to\infty} \frac{1}{n} \log \phi^s(\mathbf{j}|_n)$ exists and is independent of $\mathbf{j}$
- modified pressure:

$$P(s,\mathbf{j}) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\mathbf{i}|=n} \phi^s(\mathbf{ij}|_{\ell_n(\mathbf{i})})$$

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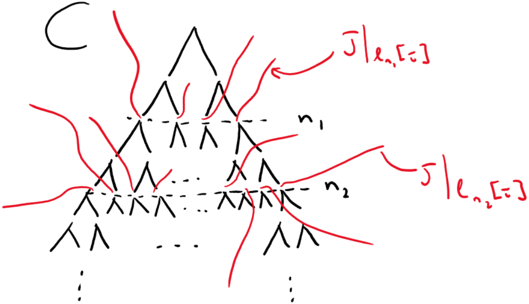
Theorem (K., Liao, Rams)

Standing assumptions. For almost all translations, for typical $\mathbf{j}$,

$$\dim_H R^*(\pi(\mathbf{j})) = s_0 : P(s_0,\mathbf{j}) = 0.$$

Path-dependent: Friday Theorem

the mass distributed according to $\phi^s \cdot \exp(Z(\mathbf{j})\ell)$ and



Moving targets: The Wednesday Theorem

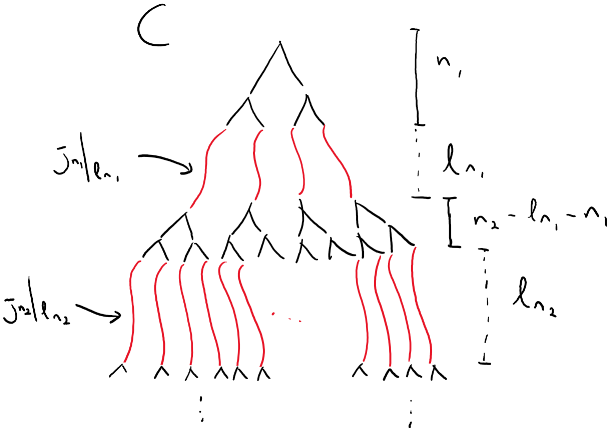
- **moving, non-path-dependent target:** $(\mathbf{j}_n)$ a sequence in Σ
- **non-path-dependent target size:** (ℓ_n)

Theorem (K., Liao, Rams)

Assume A_i satisfy a quasimultiplicativity assumption and $|A_i| < \frac{1}{2}$. For almost all translations with a strong separation condition, the dimension of the shrinking target set is given by the liminf pressure.

Moving targets: The Wednesday Theorem

the mass distributed according to ϕ^s and



And beyond

- moving AND path-dependent?
- recurrence?
- quasimultiplicativity?
- non-generic translations?