Capturing Clustering in Extreme Values

(with AC Freitas & JM Freitas)

- $T: x \mapsto 3x \bmod 1$ on $[0,1]$

  $\mu = \mathrm{Leb}$ (T-inv prob)

  Given $z \in [0,1]$, let $\varphi(x) = |x-z|^{-2}$

  $$X_k(x) = \varphi \circ T^k(x)$$

  $M_n(x) := \max\{X_0(x), \dots, X_{n-1}(x)\}$

- For $\tau > 0$ we find sequence $(u_n(\tau))_n$ $\left(= \left(\frac{1}{2}\frac{\tau}{n}\right)^{-2}\right)$ s.t.

  $$n\,\mathbb{P}(X_0 > u_n(\tau)) \to \tau$$

  For typical $z \in [0,1]$,

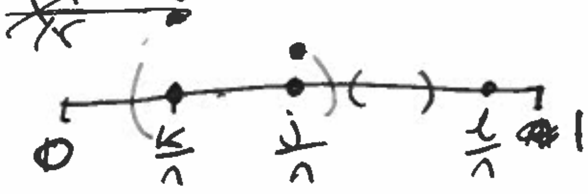
  $$\mathbb{P}(M_n \le u_n(\tau)) \to e^{-\tau}$$

Route to proof: blocking method

Record exceedances of u_n in the blocks: idea is each block should contain at most one exceedance ($D'(u_n)$) & blocks should be "independent" ← t_n

Natural tool: point process $N_n = \sum_{k=1}^{n} \delta_{\frac{k}{n}} 1_{\{X_k > u_n\}}$

Defines a random measure on $[0,1]$



Should tend to a Poisson process:

$N_n \to N$

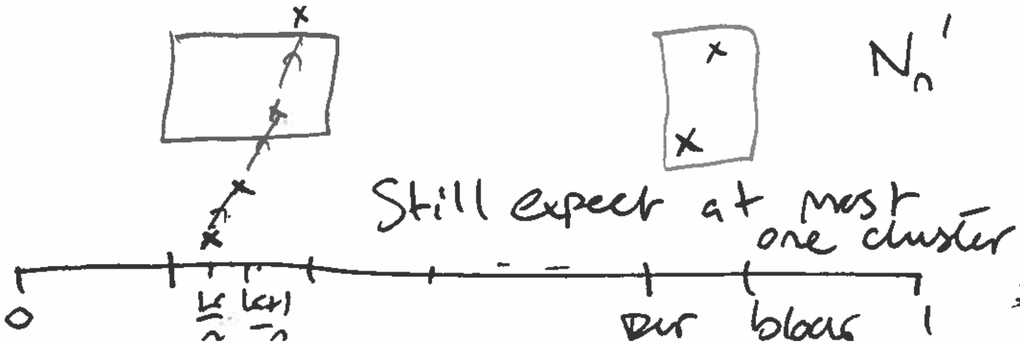
$$\left(\mathbb{P}(N(a,b)=m) = \frac{(b-a)^m}{m!} e^{-(b-a)} \right)$$

- For more info, eg Freitas-Freitas-Magalhães defined a 2D point process:

$$N_n' = \sum_{k=1}^{n} \delta_{\left(\frac{k}{n},\, u_n^{-1}(X_k)\right)}$$

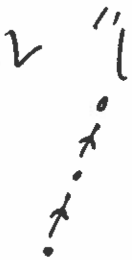
$\uparrow$ large values of $X_k \Leftrightarrow$ small values of this

- Picture alters if z is periodic, eg $z=\frac{1}{2}$, $X_k > u_n$ could be followed by $X_{k+1} > u_n$, etc.



Here N_n' tends to a compound Poisson process in the first coordinate & the second coordinate records the jumps.

But in the limit these are just "lines of jumps, pattern is lost.



- Basrak, Planinić, Soulier PTRF 18 instead associate to each block (rather than each timestep) a sequence corresponding to the X_k in the block.
- More interesting observable: $\frac{1}{8} \to \frac{3}{8} \to \frac{1}{8}$

$\varphi(x) = \left|x - \frac{1}{8}\right|^{-2} - \left|x - \frac{3}{8}\right|^{-2}$

Pattern X_k

X_{k+1}

Define

$$N_n'' = \sum_{k=1}^{k_n} \delta_{\left(\frac{k}{k_n}, \overbrace{\left(\infty, \dots, \infty, \frac{X_j}{|X_j|} u_n^{-1}(|X_j|), \frac{X_{j+1}}{|X_{j+1}|} u_n^{-1}(|X_{j+1}|), \dots, \frac{X_{j+r_n}}{|X_{j+r_n}|} u_n^{-1}(|X_{j+r_n}|), \infty, \dots\right)}^{\text{Seq for } k^{th} \text{ block}}\right)}$$

Hence the first coordinate converges to a Poisson point process & the second (an element of a sequence space) converges to a suitable sequence derived from the piling process.

- For our specific example, the limit process can be seen from $\sum_i \delta_{(T_i, U_i)}$ Poisson point process defined on $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ with intensity measure $\frac{2}{3}$ Leb $\times$ Leb & the sequences are of form

$(\dots, \infty, \infty, U_i E, -3U_i E, (-3)^2 U_i E, \dots)$

where E is iid $\mathbb{P}(E = \pm 1) = \frac{1}{2}$

Sums & Lévy processes

Setting is observables φ with (α-) heavy tails. This is a regime where ergodic sums

$$\sum_{i=0}^{n-1} X_i$$

don't satisfy a CLT.

Have to chose a scaling $(a_n)_n$ other than $\frac{1}{\sqrt{n}}$ $\left(\sim \frac{1}{n^{\frac{1}{\alpha}}}\right)$ & get convergence $\frac{1}{a_n}\sum_{i=1}^{n-1} X_i$ to α-Stable laws.

· Can also can consider

$$S_n(t) = \frac{1}{a_n}\sum_{i=1}^{\lfloor nt \rfloor} X_i$$

↑ assuming mean zero

CLT case: $S_n \to$ Brownian motion

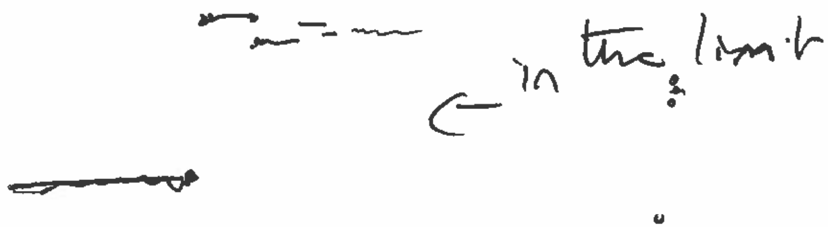
Heavy tail case associated to Lévy processes.

- In heavy-tailed setting, sums/averages $\leftrightarrow$ maxima

↑

So tools we have cover both

To see S_n we project point process to càdlàg space: "adding up the parts"



We produce decorations on cadlag space to record these patterns.