Diophantine Approximation for Systems of Linear Forms

PART 1: THE LEBESGUE MEASURE THEORY

featuring

An inhomogeneous Khintchine–Groshev Theorem without monotonicity.

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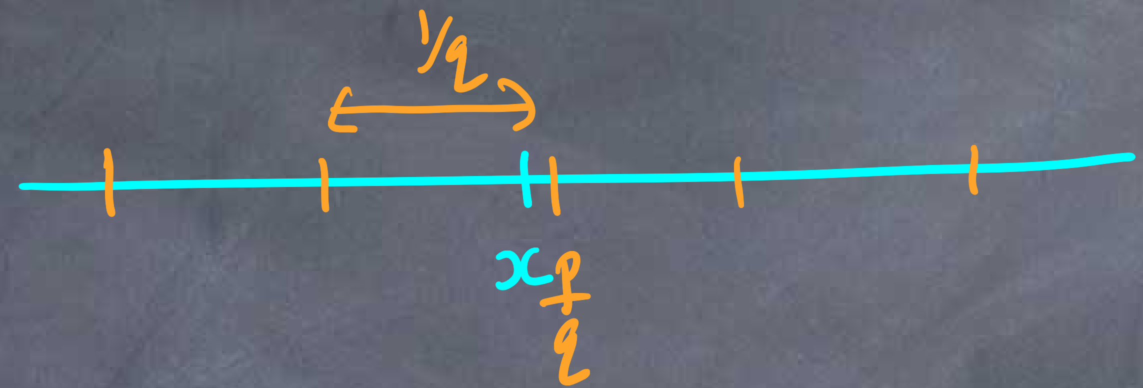
Diophantine Approximation

Ex



Given any $x \in \mathbb{R}$ and $q \in \mathbb{N}$, $\exists\ p \in \mathbb{Z}$ s.t.

$$\left| x - \frac{p}{q} \right| < \frac{1}{q}.$$

Theorem (Dirichlet, 1842) For any $x \in \mathbb{R}$, $\exists$ i.m. $q \in \mathbb{N}$, s.t.

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$$

for some $p \in \mathbb{Z}$.

Given $\psi:\mathbb{N}\to[0,\infty)$, define the ψ-well-approximable points as

$$\mathcal{A}(\psi)=\left\{x\in[0,1]: \left|x-\frac{p}{q}\right|<\frac{\psi(q)}{q} \text{ for i.m. } (p,q)\in\mathbb{Z}\times\mathbb{N}\right\}.$$

<u>Khintchine's Theorem (1924)</u> For any $\psi:\mathbb{N}\to[0,\infty)$,

$$\mathcal{L}(\mathcal{A}(\psi))=\begin{cases}0 & \text{if } \sum_{q=1}^{\infty}\psi(q)<\infty,\\ 1 & \text{if } \sum_{q=1}^{\infty}\psi(q)=\infty \text{ and } \psi \text{ is monotonic.}\end{cases}$$

<u>First Borel-Cantelli Lemma</u> Let (X,μ) be a measure space and let $(A_q)_{q\in\mathbb{N}}\subset X$ be a sequence of measurable subsets. If $\sum_{q=1}^{\infty}\mu(A_q)<\infty$, then

$$\mu\Big(\underbrace{\limsup_{q\to\infty} A_q}_{\{x\in X:\, x\in A_q \text{ for i.m. } q\in\mathbb{N}\}}\Big)=0.$$

Proof of Convergence Part of Khintchine's Theorem

For $q \in \mathbb{N}$, let $A_q = \bigcup_{p=0}^{q} B\left(\frac{p}{q}, \frac{\psi(q)}{q}\right) \cap [0,1]$.

Then $\mathcal{A}(\psi) = \limsup_{q \to \infty} A_q$.

Note that $\mathcal{L}(A_q) \leq q \times \dfrac{2\psi(q)}{q} = 2\psi(q)$.

So if $\sum_{q=1}^{\infty} \psi(q) < \infty$, then $\sum_{q=1}^{\infty} \mathcal{L}(A_q) < \infty$ and so $\mathcal{L}(\mathcal{A}(\psi)) = 0$.

Question: Do we really need monotonicity of ψ in Khintchine's Theorem?

Theorem (Duffin + Schaeffer, 1941) Khintchine's Theorem is not true in general if ψ is not monotonic.

- Let $\mathcal{A}'(\psi) = \left\{ x \in [0,1] : \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} \text{ for i.m. } (p,q) \in \mathbb{Z} \times \mathbb{N} \text{ with } \gcd(p,q) = 1 \right\}$.

Duffin - Schaeffer Conjecture (1941) For any $\psi : \mathbb{N} \to [0, \infty)$,

$$\mathcal{L}(\mathcal{A}'(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \varphi(q) \frac{\psi(q)}{q} < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} \varphi(q) \frac{\psi(q)}{q} = \infty, \end{cases}$$

where $\varphi(n) = \{ 1 \leq k \leq n : \gcd(k,n) = 1 \}$ for $n \in \mathbb{N}$.

Theorem (Koukoulopoulos + Maynard, 2020) The DSC is true.

Diophantine Approximation for Systems of Linear Forms

- Let $n, m \geq 1$ be integers.
- Let $\psi : \mathbb{N} \to [0, \infty)$.
- Write $\|\cdot\|$ for the sup. norm.
- Define

n×m matrix

$$\mathcal{A}_{n,m}(\psi) = \{ \underline{x} \in [0,1]^{nm} : \| \underline{q}\, \underline{x} - \underline{p} \| < \psi(\|\underline{q}\|) \text{ for i.m. } (\underline{p}, \underline{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \}.$$

n-dim. row   m-dim. row.

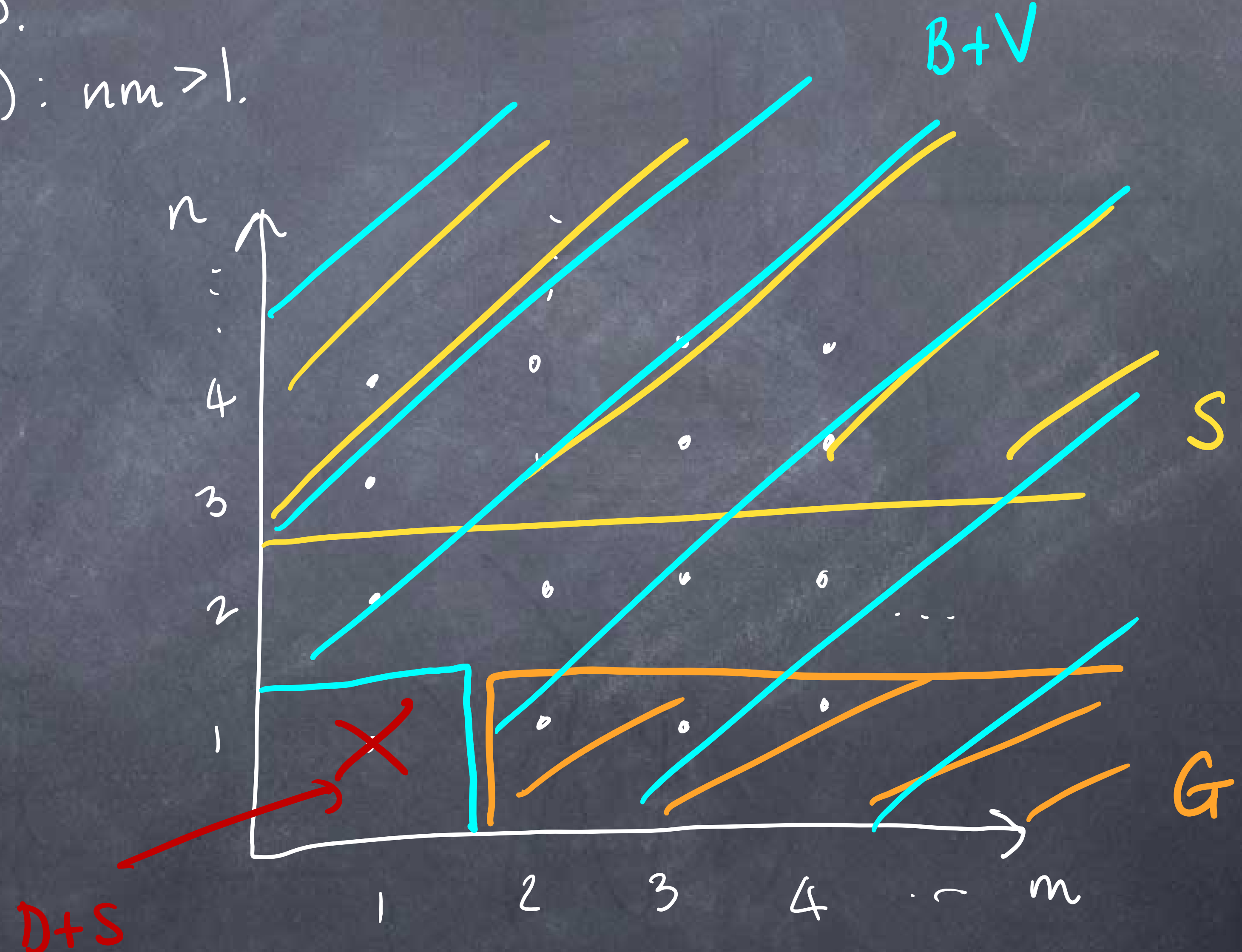
Khintchine–Groshev Theorem (1938) For any $\psi : \mathbb{N} \to [0, \infty)$,

$$\mathcal{L}(\mathcal{A}_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1} \psi(q)^m < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} q^{n-1} \psi(q)^m = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

Question Do we really need monotonicity of Ψ in the Khintchine–Groshev Theorem?

Answer YES when $n=m=1$, NO otherwise.

- ✯ Duffin + Schaeffer (1941) : $n=m=1$.
- ✯ Gallagher (1965) : $n=1$, $m\geq 2$.
- ✯ Sprindžuk (1979) : $n\geq 3$.
- ✯ Beresnevich + Velani (2010) : $nm>1$.



Inhomogeneous Khintchine-Groshev Theorem

- Let $n, m \geq 1$ be integers.
- Let $\psi : \mathbb{N} \to [0, \infty)$.
- Let $y \in \mathbb{R}^m$.
- Define

$$A^{y}_{n,m}(\psi) = \{ \underline{x} \in [0,1]^{nm} : \| \underline{q}\underline{x} - \underline{p} - \underline{y} \| < \psi(\|\underline{q}\|) \text{ for i.m. } (\underline{p}, \underline{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \}.$$

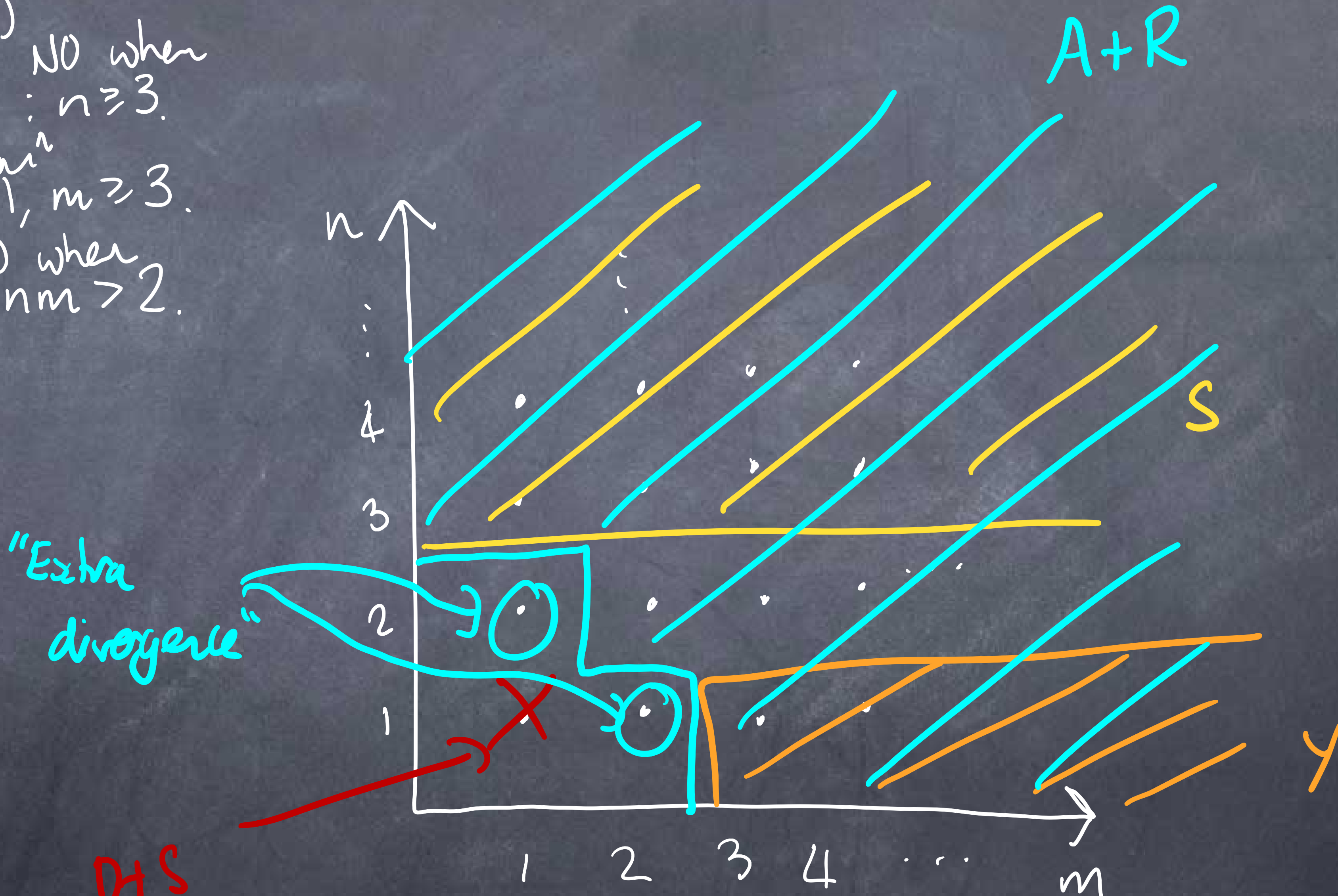
Inhomogeneous KGT (Szüsz (1958), Schmidt (1964), Sprindžuk (1979))

For any $\psi : \mathbb{N} \to [0, \infty)$ and $y \in \mathbb{R}^m$,

$$\mathcal{L}(A^{y}_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1} \psi(q)^m < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} q^{n-1} \psi(q)^m = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

Question: Do we really need monotonicity of ψ in the inhomogeneous Khintchine–Groshev Theorem?

- ★ Duffin + Schaeffer (1941): YES when $n=m=1$.
- Ramirez (2017)
- ★ Sprindžuk (1979): NO when $n \geq 3$.
- ★ Yu (2021): NO when $n=1$, $m \geq 3$.
- ★ A.–Ramirez: NO when $nm > 2$.



Comments on our proof (divergence part)

- For $\underline{q} \in \mathbb{Z}^n$, let $A^{\underline{y}}_{n,m}(\underline{q},\psi) = \{ \underline{x} \in [0,1]^{nm} : \|\underline{q}\underline{x} - \underline{p} - \underline{y}\| < \psi(\|\underline{q}\|) \text{ for } \underline{\text{some}}\ \underline{p} \in \mathbb{Z}^m \}$.
- Then $\mathcal{A}^{\underline{y}}_{n,m}(\psi) = \limsup_{\|\underline{q}\| \to \infty} A^{\underline{y}}_{n,m}(\underline{q},\psi)$.

Proof strategy

- **Aim 1 – QIA** If we could show that the sets $A^{\underline{y}}_{n,m}(\underline{q},\psi)$ were quasi-independent on average (QIA), it would follow from the Second Borel-Cantelli Lemma that $\mathcal{L}(\mathcal{A}^{\underline{y}}_{n,m}(\psi)) > 0$.

- **Aim 2 – Full measure** Lebesgue density type argument.

- **Key new idea:** Independence Inheritance.

Define a notion of ω-weak QIA (which quantifies how much independence sets have):

$$0\text{-QIA} = \text{QIA} \Rightarrow 1\text{-QIA} \Rightarrow 2\text{-QIA} \Rightarrow \ldots$$

Informal Theorem (A.–Ramírez) If the sets underlying the (n,m)-dimensional inhomogeneous Khintchine–Groshev Theorem are ω-QIA, then for $k \geq 0$, the sets underlying the $(n+k,m)$-dimensional inhomogeneous Khintchine–Groshev Theorem are $\max(0, \omega - k)$-QIA.

$(1,m)$, $m \geq 3$, QIA $\Rightarrow$ QIA for (n,m), $n \geq 1$, $m \geq 3$.

$(1,2)$, 1-QIA $\Rightarrow$ QIA for $(n,2)$, $n \geq 2$.

$(1,1)$, 2-QIA $\Rightarrow$ QIA for $(n,1)$, $n \geq 3$.

Thank You