Diophantine Approximation for Systems of Linear Forms

PART 2 : THE HAUSDORFF MEASURE THEORY

featuring

joint work with Victor Beresnevich (York, UK)

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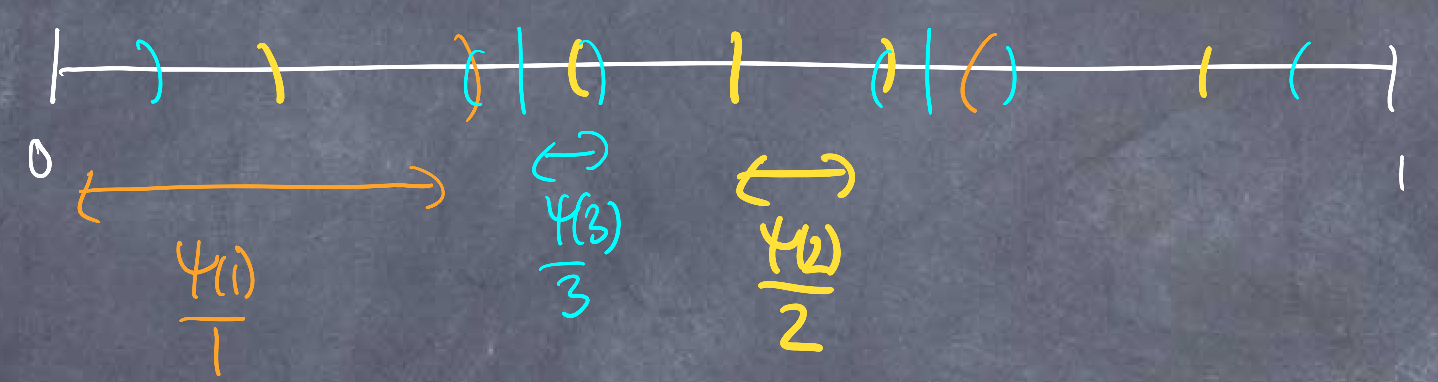
Khintchine's Theorem

Given $\psi : \mathbb{N} \to [0,\infty)$, define

$$A(\psi) = \left\{ x \in [0,1] : \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} \text{ for i.m. } (p,q) \in \mathbb{Z} \times \mathbb{N} \right\}.$$

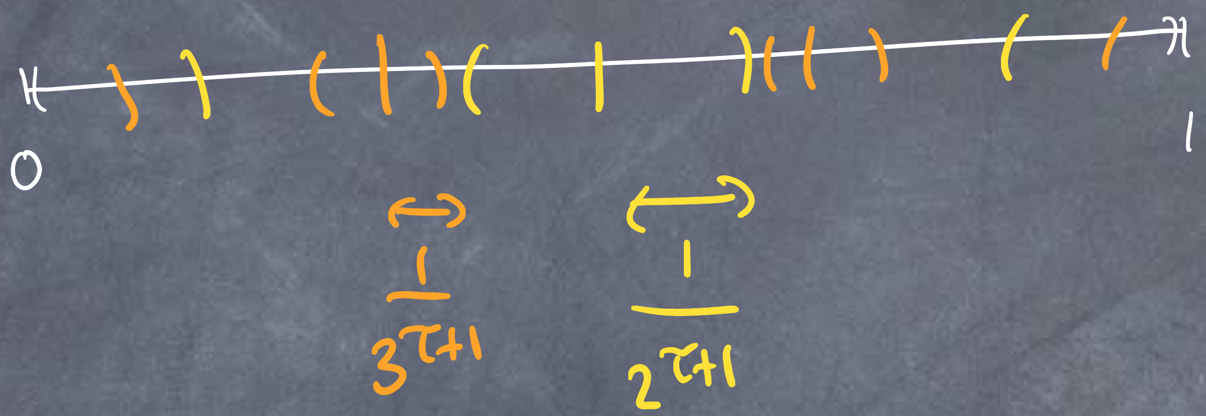


Khintchine's Theorem (1924) For any $\psi : \mathbb{N} \to [0,\infty)$,

$$\mathcal{L}(A(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q) < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} \psi(q) = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

Limitation of Lebesgue Measure

Ex Suppose $\psi(q) = q^{-\tau}$ $(\tau > 0)$ and write $W(\tau) = W(\psi)$.



Expect "size" of $W(\tau)$ to decrease as τ increases.

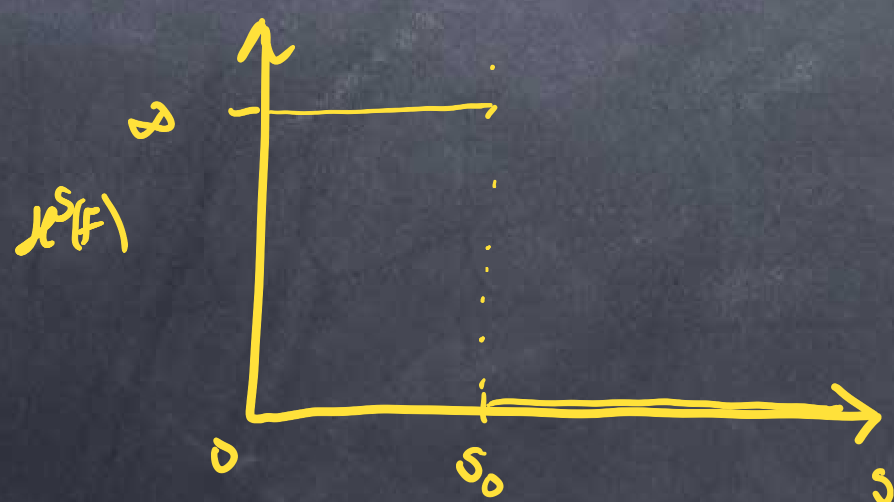
Khintchine's Theorem $\Rightarrow \mathcal{L}(W(\tau)) = 0 \quad \forall\ \tau > 1$ $\left(\text{since } \sum_{q=1}^{\infty} \psi(q) = \sum_{q=1}^{\infty} q^{-\tau}\right)$.

Theorems of Jarnik and Besicovitch

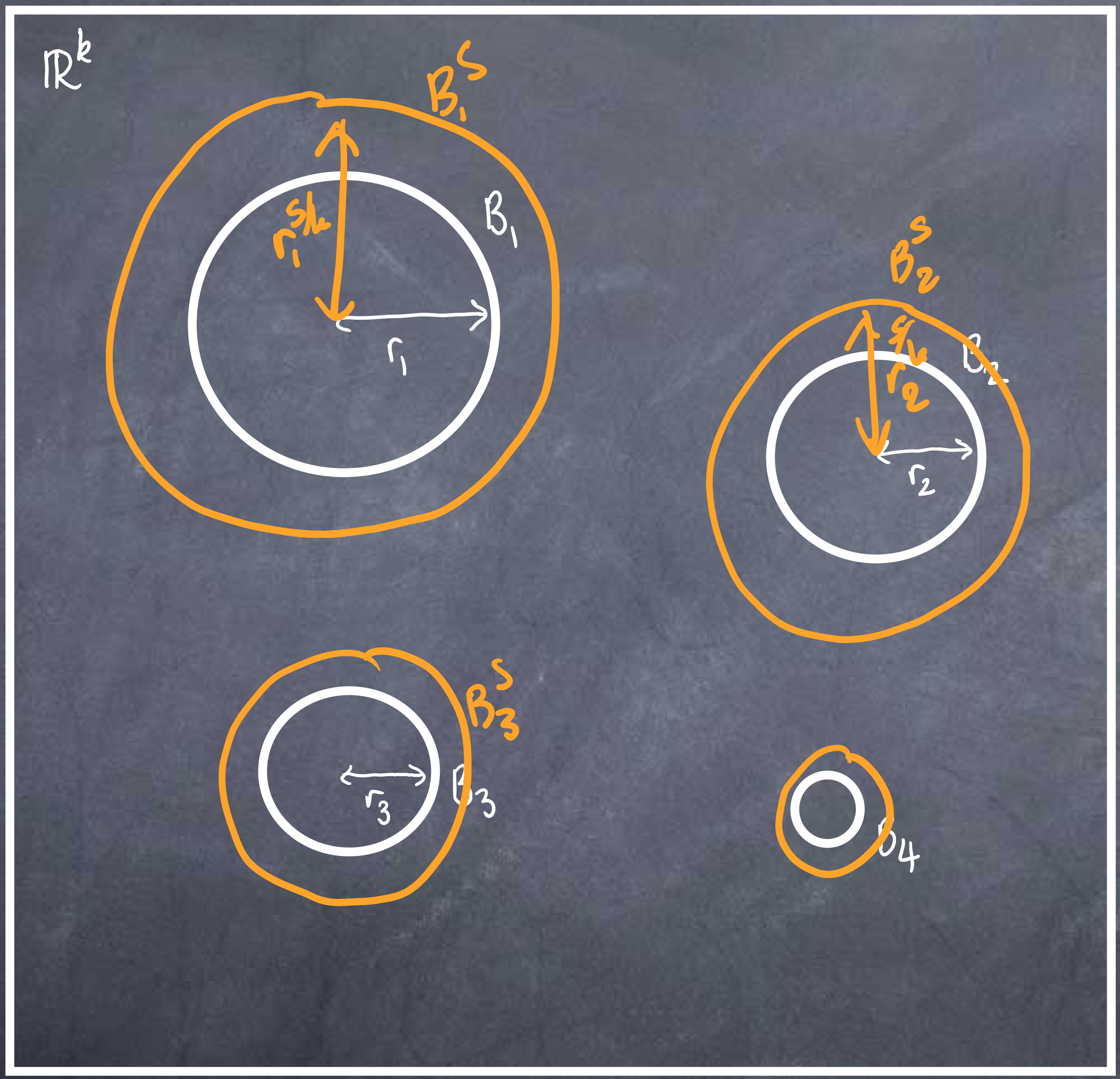
Jarnik-Besicovitch Theorem (Jarnik (1929), Besicovitch (1935)) Let $\tau > 1$. Then $\dim_H \mathcal{A}(\tau) = \dfrac{2}{\tau+1}$.

Jarnik's Theorem (1931) For $s \in (0,1)$ and $\psi : \mathbb{N} \to [0,\infty)$,

$$\mathcal{H}^s(\mathcal{A}(\psi)) = \begin{cases} 0 & \text{if } \displaystyle\sum_{q=1}^{\infty} q^{1-s}\psi(q)^s < \infty, \\ \infty & \text{if } \displaystyle\sum_{q=1}^{\infty} q^{1-s}\psi(q)^s = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$



The Mass Transference Principle



Given a ball $B = B(x, r)$ in $\mathbb{R}^k$ and $s > 0$, define $B^s = B(x, r^{s/k})$.

Mass Transference Principle (Beresnevich + Velani, 2006)

Let $\{B_j\}_{j\in\mathbb{N}}$ be a sequence of balls in $\mathbb{R}^k$ with $r(B_j)\to 0$ as $j\to\infty$. Let $s>0$ and let Ω be a ball in $\mathbb{R}^k$. Suppose that, for any ball B in Ω,

$$\mathcal{L}\left(B \cap \limsup_{j\to\infty} B_j^s\right) = \mathcal{L}(B).$$

Then, for any ball B in Ω,

$$\mathcal{H}^s\left(B \cap \limsup_{j\to\infty} B_j\right) = \mathcal{H}^s(B).$$

Some (surprising) consequences

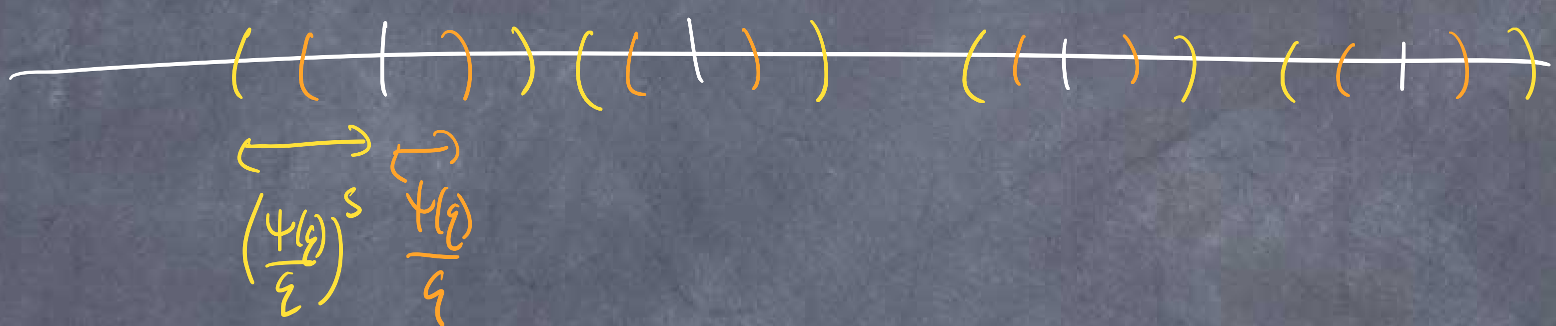
- ☆ Khintchine's Theorem $\Rightarrow$ Jarník's Theorem
- ☆ Dirichlet's Theorem $\Rightarrow$ Jarník-Besicovitch Theorem
- ☆ Duffin - Schaeffer Conjecture $\Rightarrow$ Hausdorff measure DSC.

Khintchine's Theorem ⇒ Jarník's Theorem

- Convergence: standard covering argument.

- Divergence: MTP

  - We are interested in $\mathcal{A}(\psi)$.



  - The limsup set of our "transformed" balls is $\mathcal{A}(\theta)$ where $\theta(q) = q^{1-s}\psi(q)^s$.

  - By Khintchine's Theorem, $\mathcal{L}(\mathcal{A}(\theta)) = 1$ if $\sum_{q=1}^{\infty} \theta(q) = \sum_{q=1}^{\infty} q^{1-s}\psi(q)^s = \infty$.

  - Applying the MTP with $\Omega = [0,1]$, we have for any ball B in Ω,
    $\mathcal{H}^s(B \cap \mathcal{A}(\psi)) = \mathcal{H}^s(B)$. In particular, $\mathcal{H}^s(\mathcal{A}(\psi)) = \mathcal{H}^s(\mathcal{A}(\psi) \cap [0,1]) = \mathcal{H}^s([0,1]) = \infty$.

Mass Transference Principle for Systems of Linear Forms

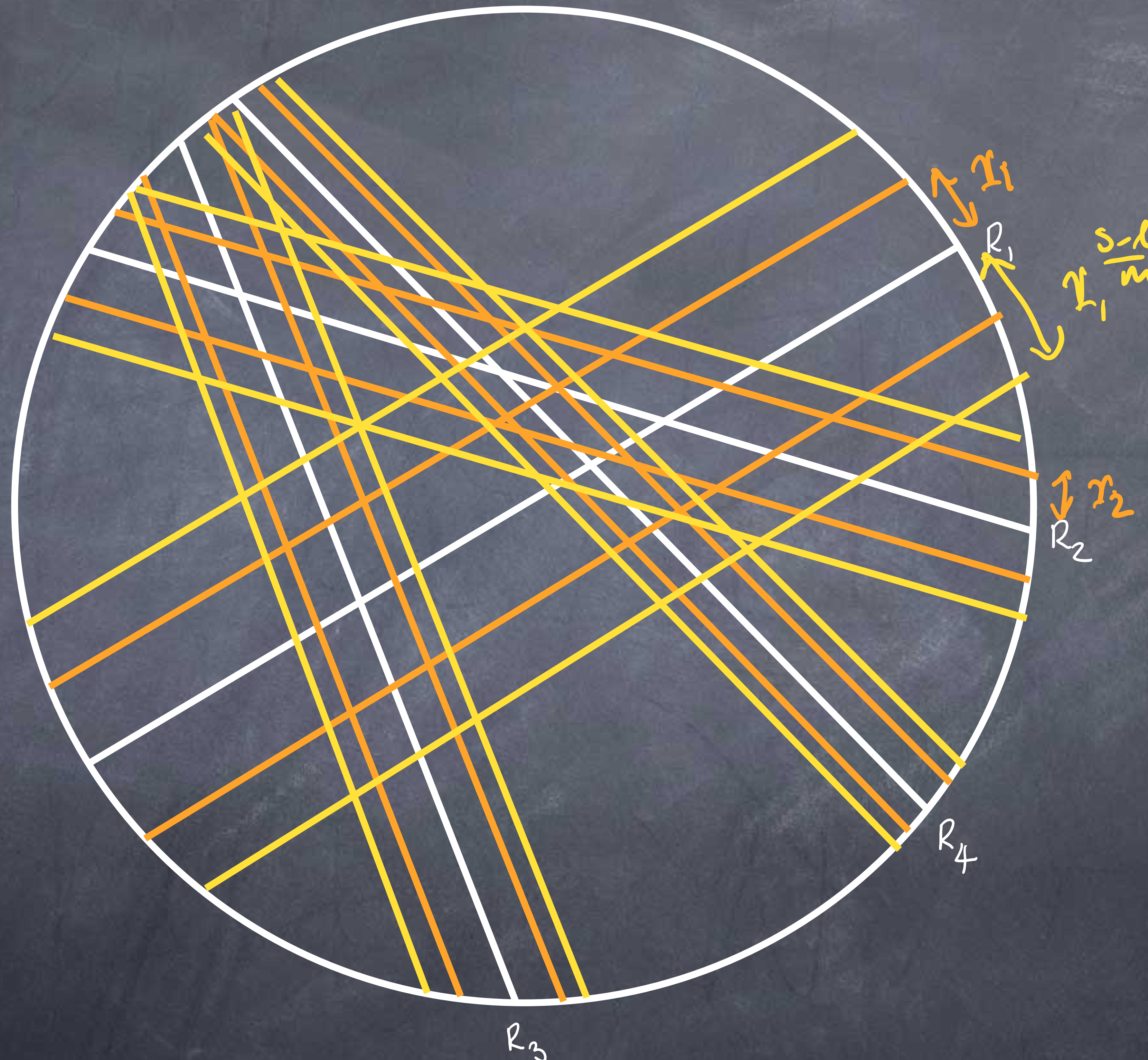
- Let $k, m \geq 1$ and $l \geq 0$ be integers s.t. $k = m + l$.
- Let $\mathcal{R} = (R_j)_{j \in \mathbb{N}}$ be a family of l-dimensional planes in $\mathbb{R}^k$.
  - For every $j \in \mathbb{N}$ and $\delta \geq 0$, define

$$\Delta(R_j, \delta) = \{\underline{x} \in \mathbb{R}^k : \operatorname{dist}(x, R_j) < \delta\}.$$

  - Let $(\Upsilon_j)_{j \in \mathbb{N}}$ be a sequence of non-negative reals s.t. $\Upsilon_j \to 0$ as $j \to \infty$.
    - Consider $\Lambda(\Upsilon) = \{\underline{x} \in \mathbb{R}^k : \underline{x} \in \Delta(R_j, \Upsilon_j)$ for i.m. $j \in \mathbb{N}\}$.
    - For $s > 0$, let $\Lambda^s(\Upsilon) = \{\underline{x} \in \mathbb{R}^k : \underline{x} \in \Delta(R_j, \Upsilon_j^{\frac{s-l}{m}})$ for i.m. $j \in \mathbb{N}\}$.

$\Lambda(\Upsilon)$

$\Lambda^s(\Upsilon)$



Theorem (A.-Beresnevich, 2018) Let $\mathcal{R}$ and $(\Upsilon_j)_{j\in\mathbb{N}}$ be as before. Let $s>0$ and let Ω be a ball in $\mathbb{R}^k$. Suppose that, for any ball B in Ω,

$$\mathcal{L}(B \cap \Lambda^s(\Upsilon)) = \mathcal{L}(B).$$

Then, for any ball B in Ω,

$$\mathcal{H}^s(B \cap \Lambda(\Upsilon)) = \mathcal{H}^s(B).$$

Remarks

1. improves upon earlier work due to Beresnevich + Velani (2006).

2. verifies a conjecture of Beresnevich + Bernik + Dodson + Velani (2009).

Applications of MTP for Systems of Linear Forms

Hausdorff Measure

Khintchine - Groshev Theorems

Diophantine Approximation for Systems of Linear Forms – the setting

- Let $n, m \geq 1$ be integers.
- Let $\psi : \mathbb{N} \to [0, \infty)$.
- Write $\|.\|$ for the sup norm.
- Let $\underline{y} \in \mathbb{R}^m$.
- Let

$$A_{n,m}^{\underline{y}}(\psi) = \left\{ \underline{x} \in [0,1]^{nm} : \|\underline{q}\underline{x} - \underline{p} - \underline{y}\| < \psi(\|\underline{q}\|) \text{ for i.m. } (\underline{p}, \underline{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \right\}.$$

Hausdorff Measure Homogeneous Khintchine–Groshev Theorem

Theorem (Beresnevich + Velani, 2010) Suppose $nm>1$ and $\psi:\mathbb{N}\to[0,\infty)$. Then

$$\mathcal{L}(\mathcal{W}^{\circ}_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m = \infty. \end{cases}$$

Hausdorff Measure Homogeneous Khintchine–Groshev Theorem

Theorem (Beresnevich + Velani, 2010) Suppose $nm>1$ and $\psi:\mathbb{N}\to[0,\infty)$. Then

$$\mathcal{L}(\mathcal{A}^{0}_{n,m}(\psi))=\begin{cases}0 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m<\infty,\\ 1 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m=\infty.\end{cases}$$

Theorem (A. + Beresnevich, 2018) Suppose $nm>1$ and $\psi:\mathbb{N}\to[0,\infty)$. For $0<s<nm$,

$$\mathcal{H}^{s}(\mathcal{A}^{0}_{n,m}(\psi))=\begin{cases}0 & \text{if } \sum_{q=1}^{\infty} q^{n+m-1}\left(\frac{\psi(q)}{q}\right)^{-m(n-1)+s}<\infty,\\ \infty & \text{if } \sum_{q=1}^{\infty} q^{n+m-1}\left(\frac{\psi(q)}{q}\right)^{-m(n-1)+s}=\infty.\end{cases}$$

Sanity check if $n=m=1$, the sum becomes $\displaystyle\sum_{q=1}^{\infty} q^{1+1-1}\left(\frac{\psi(q)}{q}\right)^{-1\times 0+s}=\sum_{q=1}^{\infty} q^{1-s}\psi(q)^s.$

Hausdorff Measure Inhomogeneous Khintchine-Groshev Theorem

Inhomogeneous Khintchine-Groshev Theorem (as of 2018) [Szüsz (1958), Schmidt (1964), Sprindžuk (1979)]

Suppose $\psi : \mathbb{N} \to [0,\infty)$ and $y \in \mathbb{R}^m$. Assume ψ is monotonic if $n=1$ or $n=2$. Then

$$\mathcal{L}(\mathcal{A}^{y}_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m = \infty. \end{cases}$$

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Theorem (A. + Beresnevich, 2018) Suppose $\psi: \mathbb{N} \to [0,\infty)$ and $y \in \mathbb{R}^m$. Let $0 < s < nm$ and if $n=1$ or $n=2$ suppose that $r\left(\frac{\psi(r)}{r}\right)^{-(n-1)+s/m}$ is monotonic. Then

$$\mathcal{H}^s(A^{y}_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n+m-1}\left(\frac{\psi(q)}{q}\right)^{-m(n-1)+s} < \infty, \\ \infty & \text{if } \sum_{q=1}^{\infty} q^{n+m-1}\left(\frac{\psi(q)}{q}\right)^{-m(n-1)+s} = \infty. \end{cases}$$

<u>Theorem (A. + Ramirez)</u> Suppose $nm > 2$, let $\psi : \mathbb{N} \to [0,\infty)$ and let $y \in \mathbb{R}^m$.

Then

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Suppose $0 < s < nm$. Then

$$\mathcal{H}^s(\mathcal{B}^y_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n+m-1}\left(\frac{\psi(q)}{q}\right)^{-m(n-1)+s} < \infty, \\ \infty & \text{if } \sum_{q=1}^{\infty} q^{n+m-1}\left(\frac{\psi(q)}{q}\right)^{-m(n-1)+s} = \infty. \end{cases}$$

Other Generalisations of the Mass Transference Principle

☆ "Original MTP" from "balls to balls" – Beresnevich + Velani (2006)

☆ MTP for systems of linear forms – Beresnevich + Velani (2006)
– A. + Beresnevich (2018)

☆ MTP for more general sets satisfying a "local scaling condition" – A. + Baker (2019)

☆ MTP from "balls to rectangles" – Wang + Wu + Xu (2015)

☆ MTP from "rectangles to rectangles" – Wang + Wu (2021)

☆ MTP from "balls to arbitrary open sets":

- Hausdorff dimension – Koivusalo + Rams (2021)
- Hausdorff measure – Zhong (2021)

☆ Multifractal MTP – Fan + Schmeling + Troubetzkoy (2013)

Thank You