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TIPES

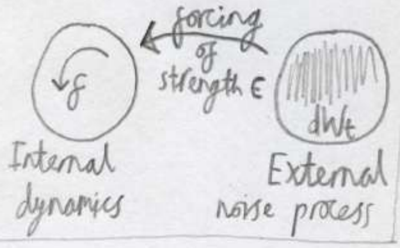
TIPPING POINTS IN THE EARTH SYSTEM

Decay of correlations ("mixing") in random dynamical systems

"noise-driven"

e.g. DS $\dot{x} = f(x)$

RDS $dx_t = f(x_t)dt + \epsilon dW_t$



Deterministic dynamical systems:

meas'ble sp. (X, Σ)

A DS on (X,Σ) is meas'ble $f: X \to X$

Defn. A prob. meas. μ on (X,Σ) is *f-invariant* if

$$\forall A \in \Sigma, \quad \mu(f^{-1}(A)) = \mu(A).$$

This $\Rightarrow \mu(f^{-n}(A)) = \mu(A)$ $\qquad f^{-n} := \underbrace{(f^n)}_{f\circ\dots\circ f}{}^{-1}$

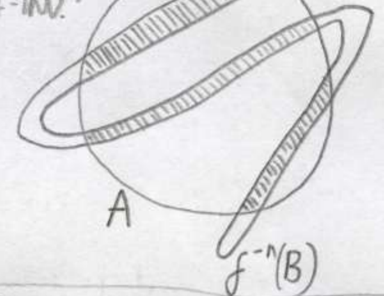
Defn. An f-inv. p.m. μ is *f-mixing* if $\forall A, B \in \Sigma$,

$$\mu(A \cap f^{-n}(B)) \longrightarrow \mu(A)\underbrace{\mu(B)}_{\mu(f^{-n}(B))} \text{ as } n \to \infty.$$

I.e. over (X, Σ, μ), A and $f^{-n}(B)$ tend towards being independent as $n \to \infty$.

$$\mathrm{LHS}(n) - \mathrm{RHS} =: \mathrm{Corr}_{f,\mu}(n; A, B).$$

This notation assumes μ is f-inv.



Ex.

$$X = \{0,1\}^{\mathbb{Z}}$$

$$\Sigma = \sigma\Big(\underbrace{\{(x_i)_{i\in\mathbb{Z}} \in X : x_j = 0\}}_{=: \pi_j^{-1}(0)} : j \in \mathbb{Z}\Big)$$

$$f((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$$

$$\mu = \left(\tfrac{1}{2}\delta_0 + \tfrac{1}{2}\delta_1\right)^{\otimes\mathbb{Z}}$$

"Two-sided Bernoulli shift on 2 symbols"

This system is mixing! (i.e. μ is f-mixing)

e.g. $A = \pi_0^{-1}(0)$

Fixing some $k \in \mathbb{N}$, $B = \pi_{-k}^{-1}(0)$.

So $f^{-n}(B) = \pi_{n-k}^{-1}(0)$.

$n > k \Rightarrow \mathrm{Corr}_{f,\mu}(n; \pi_0^{-1}(0), \pi_{-k}^{-1}(0)) = 0.$ ✓

Note: $n = k \Rightarrow \mathrm{Corr}_{f,\mu}(n; \pi_0^{-1}(0), \pi_{-k}^{-1}(0)) = \frac{1}{4}.$

So in def'n of convergence, for $\varepsilon < \frac{1}{4}$, "suff. large n" must lie within $n > k$.

$\therefore$ For fixed $A = \pi_0^{-1}(0)$, convergence not uniform across $B \in \Sigma$.

Random dynamical systems:

The RDS formalism provides a framework for extending classical dynamical systems concepts to the analysis of dynamical models that incorporate a perturbation-inducing stationary noise process.

- Prob. sp. $(\Omega, \mathcal{F}, \mathbb{P})$ ← space of sample paths of the noise
- meas'bly invertible $\theta : \Omega \to \Omega$ ← forward time-translation by one discrete time-step
  with $\mathbb{P}$ being θ-invariant ← "stationarity"
- meas'ble sp. (X, Σ)

$$\boxed{\text{Define } \pi_\Omega : \Omega \times X \to \Omega \quad \pi_\Omega(\omega, x) = \omega}$$

A RDS on (X,Σ) over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is meas'ble $\Theta : \Omega \times X \to \Omega \times X$ s.t.

$$\pi_\Omega \circ \Theta = \theta \circ \pi_\Omega .$$

→ i.e. can write $\Theta(\omega, x) = (\theta\omega, \varphi_\omega(x))$
for some ω-dependent $\varphi_\omega : X \to X$.

For $n \geq 1$, $\Theta^n(\omega, x) =: (\theta^n\omega, \underbrace{\varphi_{n,\omega}(x)}_{\varphi_{\theta^{n-1}\omega} \circ \dots \circ \varphi_\omega})$ cf. $f^n := f \circ \dots \circ f$

Typical further assumption: (X, Σ) is standard ← $\exists$ metric on X s.t.
- X is separable & complete
- $\Sigma = \mathcal{B}(X)$

"Philosophy of RDS theory": Consider concepts that only depend on the "bundle-RDS structure" of a RDS.

Bundle-RDS

- $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ as before
- meas'ble space $(Y, \mathcal{Y})$
- meas'ble $\tilde{\pi}_\Omega : Y \to \Omega$

A bundle-RDS on $(Y, \mathcal{Y}, \tilde{\pi}_\Omega)$ over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is meas'ble $\Theta : Y \to Y$ s.t.

$$\tilde{\pi}_\Omega \circ \Theta = \theta \circ \tilde{\pi}_\Omega .$$

Defn. A RDS Θ on (X, Σ) over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is equipped with the structure of a bundle-RDS by taking $(Y, \mathcal{Y}, \tilde{\pi}_\Omega) = (\Omega \times X, \mathcal{F} \otimes \Sigma, \pi_\Omega)$.

"Triviality" of RDS: A RDS Θ on (X, Σ) is trivial if it takes the form $\Theta(\omega, x) = (\theta\omega, f(x))$ for some ω-independent $f : X \to X$. "$\epsilon = 0$"

Invariance & Mixing Fix RDS Θ on standard (X, Σ) over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$.

Defn. A random prob. meas. is an Ω-indexed family $(\mu_\omega)_{\omega \in \Omega}$ of p.m. μ_ω on (X, Σ) s.t. for each $A \in \Sigma$, $\omega \mapsto \mu_\omega(A)$ is meas'ble.

Defn. A r.p.m. $(\mu_\omega)_{\omega \in \Omega}$ is Θ-invariant if for $\mathbb{P}$-a.e. $\omega \in \Omega$, we have

$$\forall A \in \Sigma, \quad \mu_\omega(\varphi_\omega^{-1}(A)) = \mu_{\theta\omega}(A).$$

This $\Rightarrow \mu_\omega(\varphi_{n,\omega}^{-1}(A)) = \mu_{\theta^n\omega}(A)$.

Defn. A random set is an Ω-indexed family $(A_\omega)_{\omega \in \Omega}$ of sets $A_\omega \in \Sigma$ s.t. $\{(\omega, x) \in \Omega \times X : x \in A_\omega\} \in \mathcal{F} \otimes \Sigma$.

Defn. A Θ-inv. r.p.m. $(\mu_\omega)_{\omega \in \Omega}$ is Θ-mixing [in an almost-sure sense] if for any random sets $(A_\omega)_{\omega \in \Omega}$, $(B_\omega)_{\omega \in \Omega}$ we have:

$$\text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \quad \mu_\omega(A_\omega \cap \varphi_{n,\omega}^{-1}(B_{\theta^n\omega})) - \mu_\omega(A_\omega) \underbrace{\mu_{\theta^n\omega}(B_{\theta^n\omega})}_{\mu_\omega(\varphi_{n,\omega}^{-1}(B_{\theta^n\omega}))} \xrightarrow[n \to \infty]{\text{as}} 0.$$

I.e. for fixed ω, over (X, Σ, μ_ω), A and $\varphi_{n,\omega}^{-1}(B_{\theta^n\omega})$ tend towards being indep. as $n \to \infty$.

The trivial cases!

Defn. A r.p.m. $(\mu_\omega)_{\omega\in\Omega}$ is <u>trivial</u> if μ_ω is indep. of ω.

We now address when a trivial r.p.m. is invariant/mixing w.r.t. a trivial RDS.

Trivial Proposition S'pose $\Theta(\omega,x) = (\theta\omega, f(x))$. Let μ be a p.m. on (X,Σ).

1. $(\mu)_{\omega\in\Omega}$ is Θ-inv. iff μ is f-inv.
2. S'pose μ is f-inv.
   $(\mu)_{\omega\in\Omega}$ is Θ-mixing iff for any random sets $(A_\omega)_{\omega\in\Omega}, (B_\omega)_{\omega\in\Omega}$ we have:
   for $\mathbb{P}$-a.e. ω, $\mathrm{Corr}_{f,\mu}(n; A_\omega, B_{\theta^n\omega}) \to 0$ as $n\to\infty$.

$(\mu)_{\omega\in\Omega}$ is Θ-mixing $\Longrightarrow$ μ is f-mixing
(just take $A_\omega = A$, $B_\omega = B$)

<u>BUT</u> may be stronger??
- the "B"-input is variable as $n\to\infty$
- decay of corr. is not uniform in "B"-input.

This is concerning: A notion of "almost sure mixing" for a noise-perturbed dynamical system <u>ought to</u> coincide with classical "mixing" of the unperturbed system if the perturbation strength is <u>literally zero</u>!

Ex. $(\Omega, \mathcal{F}, \mathbb{P}, \theta) = (X, \Sigma, \mu, f)$ = 2-sided Bern. shift on 2 symbols

$$\Theta\left((\omega_i)_{i\in\mathbb{Z}}, (x_i)_{i\in\mathbb{Z}} \right) = \left((\omega_{i+1})_{i\in\mathbb{Z}}, (x_{i+1})_{i\in\mathbb{Z}} \right).$$

Try $A_\omega = \pi_0^{-1}(0) = \{(x_i)_{i\in\mathbb{Z}} \in X : x_0 = 0\}$ (indep. of ω)

$B_\omega = \pi_{-k(\omega)}^{-1}(0) = \{(x_i)_{i\in\mathbb{Z}} \in X : x_{-k(\omega)} = 0\}$ for some $k : \Omega \to \mathbb{N}$

with $\mathbb{P}(\omega \in \Omega : \exists$ ∞-ly many $n \in \mathbb{N}$ s.t. $n \le k(\theta^n\omega)) > 0$.

n-th term: $\mathrm{Corr}_{f,\mu}\left(\boxed{n};\ \pi_0^{-1}(0),\ \pi^{-1}_{-\boxed{k(\theta^n\omega)}}(0)\right)$

$n \le k(\theta^n\omega)$ i.o.

E.g.

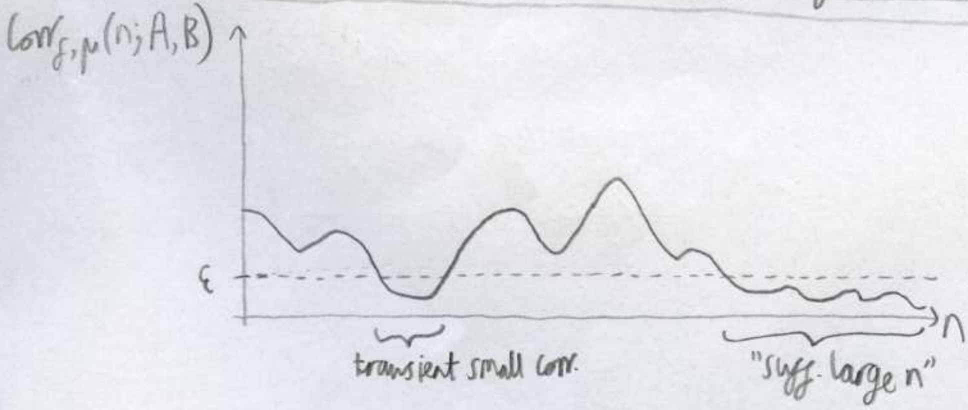
$$k\left((\omega_i)_{i\in\mathbb{Z}}\right) = 2^{2^{\min\{j\ge 0:\ \omega_j = 0\}}}$$

for $\mathbb{P}$-a.e. $(\omega_i)_{i\in\mathbb{Z}} \in \Omega$.

Using 2nd B-C Lemma, can show $\mathbb{P}(\omega\in\Omega : \ldots\ldots) = 1$. (exercise)

Q: Have I proved that $(\mu)_{\omega\in\Omega}$ is not Θ-mixing?

A: No. I've only proved that the only way $(\mu)_{\omega\in\Omega}$ could be Θ-mixing is if, paradoxically, "mixing of the RDS" "$\boxed{\text{suff. large } n}$ $\Rightarrow$ small correlation" is rescued by $\boxed{\text{transient}}$ small correlations of the classical DS.



My own invention, probably not compatible with the "philosophy of RDS theory".

Defn. Given a trivial RDS $\Theta(\omega,x) = (\theta\omega, f(x))$ on standard (X,Σ), a trivial r.p.m. $(\mu)_{\omega\in\Omega}$ is Θ-mixing in a non-paradoxical way if for any random sets $(A_\omega)_{\omega\in\Omega}$, $(B_\omega)_{\omega\in\Omega}$ we have:

for $\mathbb{P}$-a.e. ω, $\sup\limits_{m\geq n} \left| \mathrm{Corr}_{f,\mu}(m; A_\omega, B_{\theta^n\omega}) \right| \to 0$ as $n\to\infty$.

We have proved that this fails in our example!

Nonetheless:

Thm. (on MathOverflow) In our Example, $(\mu)_{\omega\in\Omega}$ is Θ-mixing!!

Two "WRONGs" have made a "RIGHT".

RIGHT: Mixing in the classical system "persists" as [almost-sure] mixing in the trivial random system.

WRONG #1: Dynamics of the noise appears non-trivially ($B_{\theta^n\omega}$) in the defn. of mixing applied to a trivial r.p.m. under a trivial RDS.

WRONG #2: In determining whether a trivial r.p.m. is mixing under a trivial RDS, a decisive role is played by transient small correlations of the classical system.

A result linking classical DS and trivial RDS

Preliminary: "UMR convergence"

Given set S and $g, g_1, g_2, g_3, \ldots : S \to \mathbb{R}$,

"$g_n \to g$ pointwise on S" means

① $\forall \varepsilon>0\ \underbrace{\forall x \in S}\ \underbrace{\exists N \in \mathbb{N} \text{ s.t.}}\ \sup_{n \geq N} |g_n(x) - g(x)| \leq \varepsilon$

OR EQUIV.

② $\forall \varepsilon>0\ \underbrace{\forall x \in S}\ \underbrace{\exists N \in \mathbb{N} \text{ s.t.}}\ \#\{n \in \mathbb{N} : |g_n(x) - g(x)| > \varepsilon\} \leq N.$

Uniformise ① $\longrightarrow$ "$g_n \to g$ uniformly on S"

i.e. $\forall \varepsilon>0\ \exists N \in \mathbb{N}$ s.t. $\sup_{n \geq N, x \in S} |g_n(x) - g(x)| \leq \varepsilon.$

Uniformise ② $\longrightarrow$ "$g_n \to g$ uniformly modulo re-ordering on S"

i.e. $\forall \varepsilon>0,\ \sup_{x \in S} \#\{n \in \mathbb{N} : |g_n(x) - g(x)| > \varepsilon\} < \infty.$

Further definitions for classical DS

(X, Σ) meas'ble $f : X \to X$ f-inv. p.m. μ

Defn. μ is uniformly f-mixing if $\forall A \in \Sigma$,

$\mu(A \cap f^{-n}(B)) \longrightarrow \mu(A)\mu(B)$ uniformly over $B \in \Sigma$.

- extremely strong Ex: One-sided Bernoulli shifts.

Defn. μ is UMR f-mixing if $\forall A \in \Sigma$,

$\mu(A \cap f^{-n}(B)) \longrightarrow \mu(A)\mu(B)$ unif. mod. re-ord. over $B \in \Sigma$.

Ex: Thm. (Ronnie Pavlov, on MathOverflow, 2022) 2-sided Bern. shift on finitely many symbols is UMR mixing! [Extended by N. to "standard" spaces of symbols.]

Thm. (N. on MathOverflow, 2022)

Assume (X,Σ) standard.

μ is UMR f-mixing iff $\forall\,(\Omega,\mathcal{F},\mathbb{P},\theta)$ as in def'n of RDS, $(\mu)_{\omega\in\Omega}$ is Θ-mixing with $\Theta(\omega,x)=(\theta\omega, f(x))$.

Proof based on B-C Lemmas.

Open questions

- Assuming (X,Σ) standard, does mixing $\Rightarrow$ UMR mixing?

  For progress so far, see mathoverflow.net/questions/418993

- Noting that the distinction between "transient dynamics" and "long-time asymptotic dynamics" for a deterministic autonomous DS is somewhat fuzzy,

  $\boxed{\text{E.g. given surjective } f\colon X\to X,\ f^n \text{ for large } n \text{ determines } f.}$

  is there an equivalent (even if more complex) defn. of mixing for RDS that elucidates a clearer physical interpretation and gets round the "two WRONGs"?