L^q-spectra of self-affine measures: closed forms, counterexamples, and split binomial sums

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Based on joint work with Jonathan Fraser, Ian Morris and Han Yu

Iterated Function Systems

A common way to generate fractals is through the use of *iterated function systems* (IFS), which are just a collection of contractions $S_1, \ldots, S_m$ on some closed set $D \subset \mathbb{R}^n$.

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A common way to generate fractals is through the use of *iterated function systems* (IFS), which are just a collection of contractions $S_1,\ldots,S_m$ on some closed set $D \subset \mathbb{R}^n$.

By contractions we mean that for each i
$|S_i(x) - S_i(y)| \leq C_i|x - y|$ for each $x, y \in D$, where $0 \leq C_i < 1$.

Iterated Function Systems

Associated to each IFS is a unique invariant set called the *attractor*.

Theorem (Hutchinson, 1986)

Let $S_1,\ldots,S_m$ be an IFS. Then there exists a unique non-empty compact set F satisfying

$$F = \bigcup_{i=1}^{m} S_i(F)$$

We call this set the *attractor* of the IFS.

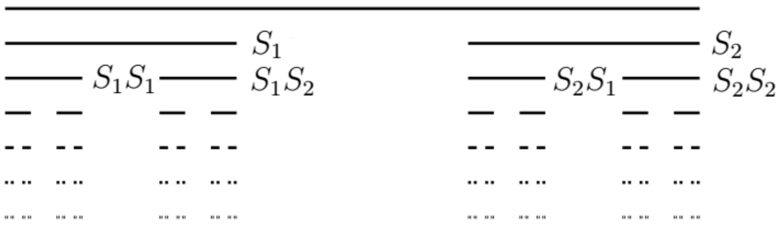
Self-Similar Sets

If the contractions in our IFS are similarities, so that for each i $|S_i(x) - S_i(y)| = r_i|x - y|$ for each $x, y \in D$, where $0 \leq r_i < 1$, then we call the corresponding attractor F a *self-similar set*.

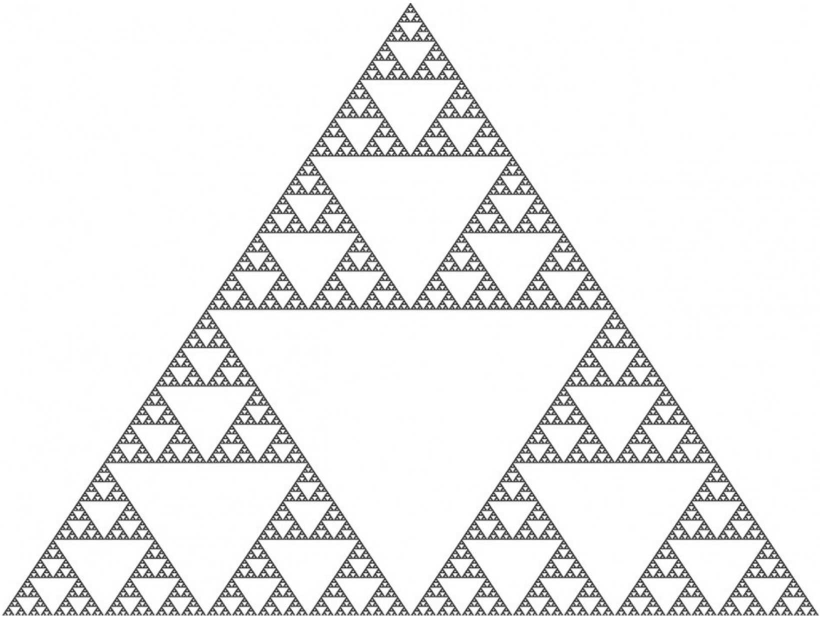
Middle Third Cantor Set

Middle Third Cantor Set

$$S_1(x) = \frac{x}{3},\ S_2(x) = \frac{x}{3} + \frac{2}{3}$$



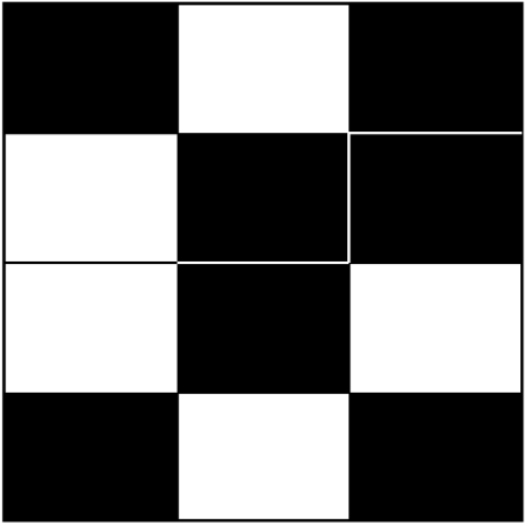
Sierpinski Triangle



Self-Affine Sets

If the contractions in our IFS are affine maps, so that for each i $S_i(x) = A_i(x) + t_i$ where A_i is a linear transformation on $\mathbb{R}^n$ and t_i is a vector in $\mathbb{R}^n$, then we call the corresponding attractor F a *self-affine set*.

Bedford-McMullen Carpets



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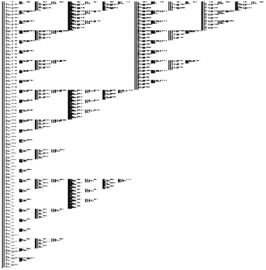
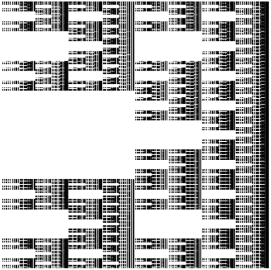
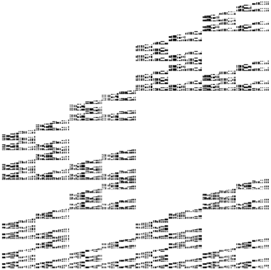
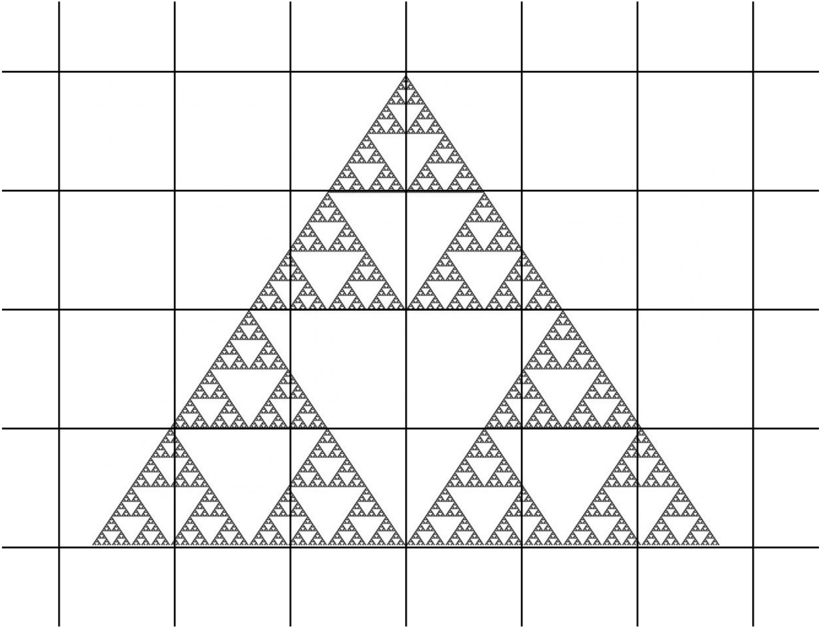




Box Dimension

We often study fractals using different notions of dimension. These are quantities which typically return the “expected” value for traditional Euclidean shapes (e.g. 1 for lines, 2 for squares etc.) but may take non-integer values for more irregular constructions. They help to quantify how “rough” the fractal is.

One example of such a dimension is box dimension, which has been extensively studied.

Box Dimension



Box Dimension

Definition

Let $F \subseteq \mathbb{R}^n$ and let $N_\delta(F)$ denote the number of δ-mesh cubes that intersect F. We define the upper and lower box dimensions of F by

$$\overline{\dim}_{\mathrm{B}} F = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

and

$$\underline{\dim}_{\mathrm{B}} F = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

respectively.

If they coincide then we define their common value to be the box dimension of F.

Hutchinson's Formula

In "nice cases" the box dimension of a self-similar set is given by the unique s satisfying

$$\sum_{i=1}^{m} r_i^s = 1.$$

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In "nice cases" the box dimension of a self-similar set is given by the unique s satisfying

$$\sum_{i=1}^{m} r_i^s = 1.$$

For the middle third Cantor set $r_1 = r_2 = 1/3$, so this formula gives the box dimension to be $s = \log 2 / \log 3 \approx 0.631$.

Hutchinson's Formula

Note that this expression for s is a **closed form** expression, i.e. it contains a finite number of standard operations (such as addition and exponentiation) but does not involve limits, derivatives, integrals etc.

Multifractals

What is a multifractal? You can think of a multifractal as being a measure which is distributed in a highly irregular way.

Self-Similar Measures

Definition (Self-similar measure)

Suppose we have a self-similar set F given by the IFS $\{S_i\}_{i\in\mathcal{I}}$, and a probability vector $\{p_i\}_{i\in\mathcal{I}}$ with each $p_i \in (0,1)$. Then there is a unique probability measure μ satisfying

$$\mu = \sum_{i\in\mathcal{I}} p_i \ \mu \circ S_i^{-1}$$

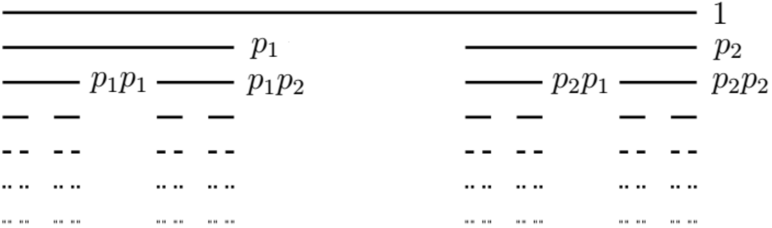
which we call the *self-similar measure* associated to $\{S_i\}_{i\in\mathcal{I}}$ and $\{p_i\}_{i\in\mathcal{I}}$.

Self-Similar Measures

$p_1 + p_2 = 1$

$0 < p_1, p_2 < 1$

$p_1 : p_2$



Self-Similar Measures

Self-similar measures can also be thought of as being pushforward Bernoulli measures.

Self-Affine Measures

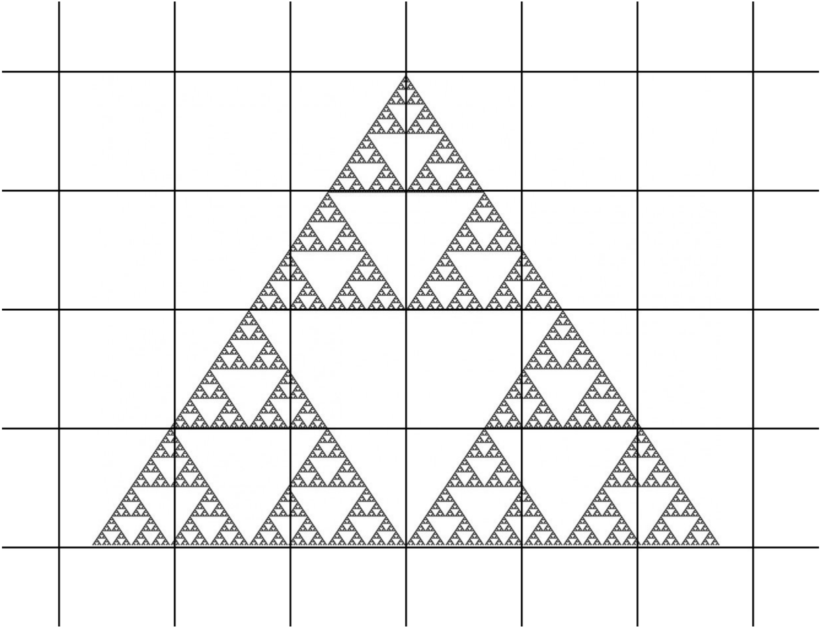
Definition (Self-affine measure)

Suppose we have a self-affine set F given by the IFS $\{S_i\}_{i\in\mathcal{I}}$, and a probability vector $\{p_i\}_{i\in\mathcal{I}}$ with each $p_i \in (0,1)$. Then there is a unique probability measure μ satisfying

$$\mu = \sum_{i\in\mathcal{I}} p_i \; \mu \circ S_i^{-1}$$

which we call the *self-affine measure* associated to $\{S_i\}_{i\in\mathcal{I}}$ and $\{p_i\}_{i\in\mathcal{I}}$.

The L^q-Spectrum



The L^q-Spectrum

Let μ be a probability measure, $\delta > 0$ and write $\mathcal{D}_\delta$ to denote the set of closed cubes in a δ mesh on $\mathbb{R}^n$ centred at the origin with positive measure. Write

$$\mathcal{D}_\delta^q(\mu) = \sum_{Q \in \mathcal{D}_\delta} \mu(Q)^q$$

where the sum is taken over those Q such that $\mu(Q) > 0$.

The L^q-Spectrum

Definition

If μ is a Borel probability measure on $\mathbb{R}^n$ then for $q \geq 0$ the upper and lower L^q spectrum of μ are defined to be

$$\overline{\tau}_\mu(q) = \overline{\lim}_{\delta \to 0} \frac{\log \mathcal{D}_\delta^q(\mu)}{-\log \delta}$$

and

$$\underline{\tau}_\mu(q) = \underline{\lim}_{\delta \to 0} \frac{\log \mathcal{D}_\delta^q(\mu)}{-\log \delta}$$

respectively.

If they coincide then we define their common value to be the L^q spectrum of μ.

The L^q-Spectrum

The L^q-spectrum has several useful properties.

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Firstly, the box dimension of the set μ is supported on is given by $\tau_\mu(0)$.

The L^q-Spectrum

Secondly, if $\tau_\mu(q)$ is differentiable at $q = 1$ then $-\tau'_\mu(1)$ gives the Hausdorff dimension of the *measure* μ.

This is defined by

$$\dim_H \mu = \inf\{\dim_H A : \mu(A) = 1\}.$$

The L^q-Spectrum

Finally, the L^q-spectrum can be used in some situations to calculate the *fine multifractal spectrum* of μ.

Local Dimension

We define the local dimension of μ at x by

$$\dim_{\mathrm{loc}}(x) := \lim_{r \to 0} \frac{\log(\mu(B(x,r)))}{\log r}.$$

The L^q-Spectrum

For $\alpha \geq 0$ we define

$$F_\alpha = \{x \in \mathbb{R}^n : \dim_{\mathrm{loc}}(x) = \alpha\}.$$

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For $\alpha \geq 0$ we define

$$F_\alpha = \{x \in \mathbb{R}^n : \dim_{\mathrm{loc}}(x) = \alpha\}.$$

We then define the fine multifractal spectrum of μ to be the function

$$f_H(\alpha) = \dim_H(F_\alpha).$$

The L^q-Spectrum

How can the L^q-spectrum help us understand the fine multifractal spectrum?

Begin by defining the *Legendre transform* of $\tau_\mu(q)$ by

$$f(\alpha) = \inf_{q \in \mathbb{R}} \{\tau_\mu(q) + \alpha(q)\}.$$

The L^q-Spectrum

How can the L^q-spectrum help us understand the fine multifractal spectrum?

Begin by defining the *Legendre transform* of $\tau_\mu(q)$ by

$$f(\alpha) = \inf_{q \in \mathbb{R}} \{\tau_\mu(q) + \alpha(q)\}.$$

Then we always have

$$f_H(\alpha) \leq f(\alpha).$$

In many "nice" cases equality holds in the above (i.e. $f_H(\alpha) = f(\alpha)$). If this happens we say that the *multifractal formalism* holds.

The L^q-Spectrum

If μ is a "nice" self-similar measure with contractions r_i and probabilities p_i then the L^q-spectrum of μ is given by the function $\beta : \mathbb{R} \to \mathbb{R}$, where β is defined by

$$\sum_{i \in \mathcal{I}} p_i^q r_i^{\beta(q)} = 1.$$

The L^q-Spectrum

Note that this expression for $\beta(q)$ is a **closed form** expression, i.e. it contains a finite number of standard operations (such as addition and exponentiation) but does not involve limits, derivatives, integrals etc.

Diagonal Systems

Definition (Diagonal System)

We say a self-affine IFS $\{S_i\}_{i \in \mathcal{I}}$ is a *diagonal system* if each map is of the form $S_i(x, y) = T_i(x, y) + t_i$, where T_i can be written in matrix form as

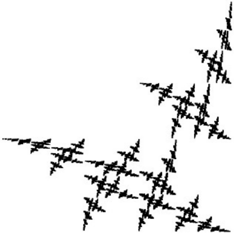
$$T_i(x, y) = \begin{pmatrix} \pm c_i & 0 \\ 0 & \pm d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

with $c_i, d_i \in (0, 1)$ and $t_i \in \mathbb{R}^2$ is a translation vector

Diagonal Systems

We shall study the L^q-spectrum of self-affine measures supported on the attractors of diagonal systems.

Box-Like Self-Affine Sets



Rectangular Open Set Condition

We will assume that all of our sets satisfy the rectangular open set condition.

Definition (Rectangular Open Set Condition)

We say an IFS acting on $\mathbb{R}^2$ satisfies the *Rectangular Open Set Condition* (ROSC) if there exists a non-empty open rectangle $R = (a, b) \times (c, d) \subset \mathbb{R}^2$ such that $\{S_i(R)\}_{i \in \mathcal{I}}$ are pairwise disjoint subsets of R.

Calculating the L^q-spectrum

In order to calculate the L^q-spectrum $\tau_\mu(q)$ of self-affine measures supported on diagonal systems, Fraser introduced what he termed a *q-modified singular value function*.

Some Notation

Let $\mathcal{I}^* = \bigcup_{k \geq 1} \mathcal{I}^k$ denote the set of all finite sequences with entries in $\mathcal{I}$. For $\mathbf{i} = (i_1, \ldots, i_k) \in \mathcal{I}^*$ let $S_{\mathbf{i}} = S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_k}$ and let $p(\mathbf{i}) = p_{i_1} p_{i_2} \cdots p_{i_k}$.

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Also write $\alpha_1(\mathbf{i}) \geq \alpha_2(\mathbf{i})$ for the singular values of the linear part of $S_{\mathbf{i}}$ and write $c(\mathbf{i}) = c_{i_1} c_{i_2} \cdots c_{i_k}$ and $d(\mathbf{i}) = d_{i_1} d_{i_2} \cdots d_{i_k}$.

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In particular, for all $\mathbf{i} = (i_1, \dots, i_k) \in \mathcal{I}^*$, $\alpha_1(\mathbf{i}) = \max\{c(\mathbf{i}), d(\mathbf{i})\}$ and $\alpha_2(\mathbf{i}) = \min\{c(\mathbf{i}), d(\mathbf{i})\}$.

Projections

We define the projection maps $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$.

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For $q \geq 0$ we define

$$\tau_1(q) = \tau_{\pi_1(\mu)}(q)$$

and

$$\tau_2(q) = \tau_{\pi_2(\mu)}(q).$$

(It may be shown that both of these L^q-spectra exist).

The q-modified singular value function

Definition (Fraser)

For $s \in \mathbb{R}$ and $q \geq 0$ we define the *q-modified singular value function* $\psi^{s,q} : \mathcal{I}^* \to (0, \infty)$ by

$$\psi^{s,q}(\mathbf{i}) = p(\mathbf{i})^q \; \alpha_1(\mathbf{i})^{\tau_{\mathbf{i}}(q)} \; \alpha_2(\mathbf{i})^{s-\tau_{\mathbf{i}}(q)}$$

A Pressure Function

We may now define a function $P : \mathbb{R} \times [0, \infty) \to [0, \infty)$ by

$$P(s, q) = \lim_{k \to \infty} \left(\sum_{\mathbf{i} \in \mathcal{I}^k} \psi^{s,q}(\mathbf{i}) \right)^{1/k}$$

(the fact this limit exists follows from some technical results and properties of sub- and super- multiplicative sequences).

The Function $\gamma(q)$

Lemma (Fraser)

For each $q \geq 0$ there is a unique $s \geq 0$ such that $P(s, q) = 1$.

We may therefore define a function $\gamma : [0, \infty) \to \mathbb{R}$ by $P(\gamma(q), q) = 1$. The importance of this function is the following theorem of Fraser.

The Function $\gamma(q)$

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Theorem (Fraser)

Let μ be a self-affine measure supported on a diagonal system satisfying the ROSC and let $q \geq 0$. Then

$$\tau_\mu(q) = \gamma(q).$$

The Function $\gamma(q)$

You can think of $\gamma(q)$ as an analogue of $\beta(q)$, which we saw earlier as the function which gives the L^q-spectrum of "nice" self-similar measures.

$$\sum_{i \in \mathcal{I}} p_i^q r_i^{\beta(q)} = 1.$$

Closed Form Expressions

Unfortunately we have no “nice” way to calculate $\gamma(q)$. Ideally we would like to be able to find a **closed form** expression for γ. Fraser proved that this is possible, but only in certain cases.

The Functions γ_A and γ_B

We can define functions $\gamma_A, \gamma_B : [0, \infty) \to \mathbb{R}$ by

$$\sum_{i \in \mathcal{I}} p_i^q \; c_i^{\tau_1(q)} \; d_i^{\gamma_A(q) - \tau_1(q)} = 1$$

and

$$\sum_{i \in \mathcal{I}} p_i^q \; d_i^{\tau_2(q)} \; c_i^{\gamma_B(q) - \tau_2(q)} = 1.$$

A Closed Form Expression

There are two possibilities.

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Theorem (Fraser)

Let $q \geq 0$*. If* $\max\{\gamma_A(q), \gamma_B(q)\} \leq \tau_1(q) + \tau_2(q)$*, then*

$$\gamma(q) = \max\{\gamma_A(q), \gamma_B(q)\}.$$

A Closed Form Expression

Theorem (Fraser)

If $\min\{\gamma_A(q), \gamma_B(q)\} \geq \tau_1(q) + \tau_2(q)$, *then*

$$\gamma(q) \leq \min\{\gamma_A(q), \gamma_B(q)\}$$

A Closed Form Expression

Theorem (Fraser)

If $\min\{\gamma_A(q), \gamma_B(q)\} \geq \tau_1(q) + \tau_2(q)$, *then*

$$\gamma(q) \leq \min\{\gamma_A(q), \gamma_B(q)\}$$

with equality holding if either

$$\sum_{i \in \mathcal{I}} p_i^q \; c_i^{\tau_1(q)} \; d_i^{\gamma_A(q) - \tau_1(q)} \log(c_i/d_i) \geq 0$$

or

$$\sum_{i \in \mathcal{I}} p_i^q \; d_i^{\tau_2(q)} \; c_i^{\gamma_B(q) - \tau_2(q)} \log(d_i/c_i) \geq 0.$$

A Question

Question (Fraser)

If $\min\{\gamma_A(q), \gamma_B(q)\} \geq \tau_1(q) + \tau_2(q)$ *and neither*

$$\sum_{i \in \mathcal{I}} p_i^q \; c_i^{\tau_1(q)} \; d_i^{\gamma_A(q) - \tau_1(q)} \; \log(c_i/d_i) \geq 0$$

nor

$$\sum_{i \in \mathcal{I}} p_i^q \; d_i^{\tau_2(q)} \; c_i^{\gamma_B(q) - \tau_2(q)} \; \log(d_i/c_i) \geq 0$$

are satisfied, is it still true that

$$\gamma(q) = \min\{\gamma_A(q), \gamma_B(q)\}?$$

A Family of Counterexamples

Theorem (Fraser, L, Morris, Yu)

Let c, d be such that $c > d > 0$ and $c + d \leq 1$. Let μ be the self-affine measure defined by the probability vector $(1/2, 1/2)$ and the diagonal system consisting of the two maps, S_1 and S_2, where

$$S_1(x,y) = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } S_2(x,y) = \begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1-d \\ 1-c \end{pmatrix}.$$

Then, for $q > 1$,

$$\gamma(q) < \min\{\gamma_A(q), \gamma_B(q)\}.$$

Split Binomial Sums

A key lemma concerns the binomial expansion of $(1+x)^k$.

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Lemma (Fraser, L, Morris, Yu)

Let $x > 1$, *then*

$$\lim_{k\to\infty}\left(\frac{\sum_{i=\lceil k/2\rceil}^{k}\binom{k}{i}x^i}{\sum_{i=0}^{\lfloor k/2\rfloor}\binom{k}{i}x^i}\right)^{\frac{1}{k}} = \frac{1+x}{2\sqrt{x}} > 1$$

where the limit is taken along odd integers k.

The Proof (Sketch)

Due to the relative simplicity of the maps S_1, S_2 it is straightforward to show that $\tau_1(q) = \tau_2(q) = \gamma_A(q) = \gamma_B(q)$. We shall denote this common value by s.

The Proof (Sketch)

We may write

$$P(s,q) = \lim_{k\to\infty}\left(\sum_{\mathbf{i}\in\mathcal{I}^k}\psi^{s,q}(\mathbf{i})\right)^{1/k} = \lim_{k\to\infty}\left(X_k^q + Y_k^q\right)^{1/k},$$

where

$$X_k^q = \sum_{i=0}^{\lfloor k/2\rfloor}\binom{k}{i}2^{-kq}\left(c^{k-i}d^i\right)^s$$

and

$$Y_k^q = \sum_{i=\lceil k/2\rceil}^{k}\binom{k}{i}2^{-kq}\left(d^{k-i}c^i\right)^s.$$

The Proof (Sketch)

By the binomial theorem and the definition of $s = \gamma_A(q)$,

$$\sum_{i=0}^{k} \binom{k}{i} 2^{-kq} \left(c^{k-i} d^i\right)^s = \left(2^{-q} c^{\gamma_A(q)} + 2^{-q} d^{\gamma_A(q)}\right)^k = 1^k = 1.$$

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By the binomial theorem and the definition of $s = \gamma_A(q)$,

$$\sum_{i=0}^{k} \binom{k}{i} 2^{-kq} \left(c^{k-i} d^i\right)^s = \left(2^{-q} c^{\gamma_A(q)} + 2^{-q} d^{\gamma_A(q)}\right)^k = 1^k = 1.$$

It can then be shown that

$$\frac{X_k^q}{1 - X_k^q} = \frac{\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{i} \left((d/c)^s\right)^i}{\sum_{i=\lceil k/2 \rceil}^{k} \binom{k}{i} \left((d/c)^s\right)^i}.$$

The Proof (Sketch)

By our binomial result,

$$\left(\frac{X_k^q}{1 - X_k^q}\right)^{1/k} \to \frac{2(d/c)^{s/2}}{(d/c)^s + 1} =: \delta \in (0, 1)$$

as $k \to \infty$.

The Proof (Sketch)

This implies that $\left(X_k^q\right)^{1/k} \to \delta$ as $k \to \infty$, and by similar reasoning we can also deduce that $\left(Y_k^q\right)^{1/k} \to \delta$ as $k \to \infty$.

The Proof (Sketch)

Therefore,

$$P(s,q) = \lim_{k\to\infty} \left(X_k^q + Y_k^q\right)^{1/k} = \delta < 1$$

and by definition of $P(t,q)$ and $\gamma(q)$

$$P(\gamma(q),q) = 1 > \delta = P(s,q).$$

The Proof (Sketch)

Therefore,

$$P(s,q) = \lim_{k\to\infty} \left(X_k^q + Y_k^q\right)^{1/k} = \delta < 1$$

and by definition of $P(t,q)$ and $\gamma(q)$

$$P(\gamma(q),q) = 1 > \delta = P(s,q).$$

Since $P(t,q)$ is decreasing in t this implies $\gamma(q) < s = \gamma_A(q) = \gamma_B(q)$, so $\gamma(q) < \min\{\gamma_A(q), \gamma_B(q)\}$.

Phase Transitions

Theorem (Fraser, L, Morris, Yu)

There exists a planar self-affine measure μ defined by an IFS satisfying the rectangular open set condition (ROSC) such that τ_μ, the L^q-spectrum of μ, is differentiable at $q = 1$ but not analytic in any neighbourhood of $q = 1$.

New Closed Form Lower Bounds

Theorem (Fraser, L, Morris, Yu)

Let μ be a self-affine measure generated by a diagonal system and let $q \geq 0$. Then

$$\gamma(q) \geq \max\{L_A(q), L_B(q)\}$$

where $L_A(q) = \gamma_A(q) -$

$$\left(\Big(\gamma_A(q) - \tau_1(q) - \tau_2(q)\Big) \frac{\sum_{i \in \mathcal{I}} p_i^q \; c_i^{\tau_1(q)} \; d_i^{\gamma_A(q) - \tau_1(q)} \log(c_i/d_i)}{\sum_{i \in \mathcal{I}} p_i^q \; c_i^{\tau_1(q)} \; d_i^{\gamma_A(q) - \tau_1(q)} \log(c_i)} \right)^+$$

and $L_B(q) = \gamma_B(q) -$

$$\left(\Big(\gamma_B(q) - \tau_1(q) - \tau_2(q)\Big) \frac{\sum_{i \in \mathcal{I}} p_i^q \; d_i^{\tau_2(q)} \; c_i^{\gamma_B(q) - \tau_2(q)} \log(d_i/c_i)}{\sum_{i \in \mathcal{I}} p_i^q \; d_i^{\tau_2(q)} \; c_i^{\gamma_B(q) - \tau_2(q)} \log(d_i)} \right)^+ .$$

Thank You For Listening!