Certification of Almost Global Phase Synchronization of All-To-All Coupled Phase Oscillators

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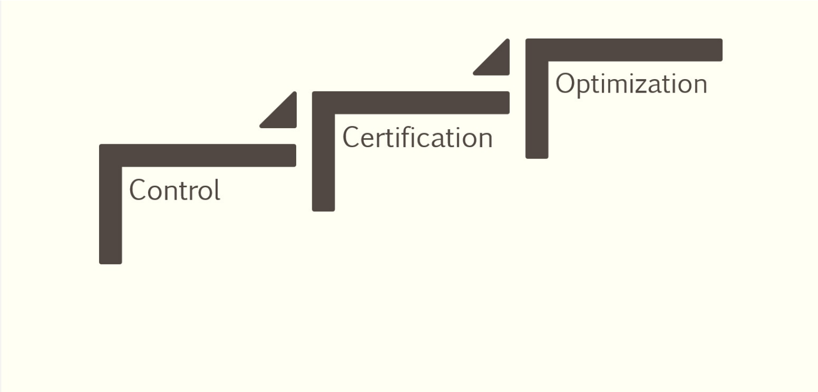
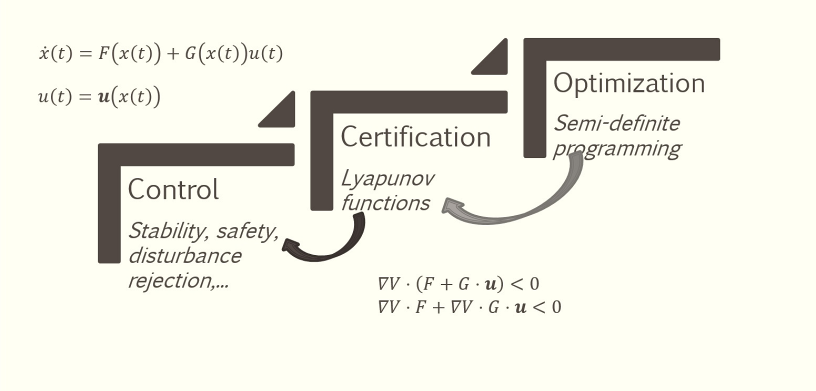
Certification of Attractors

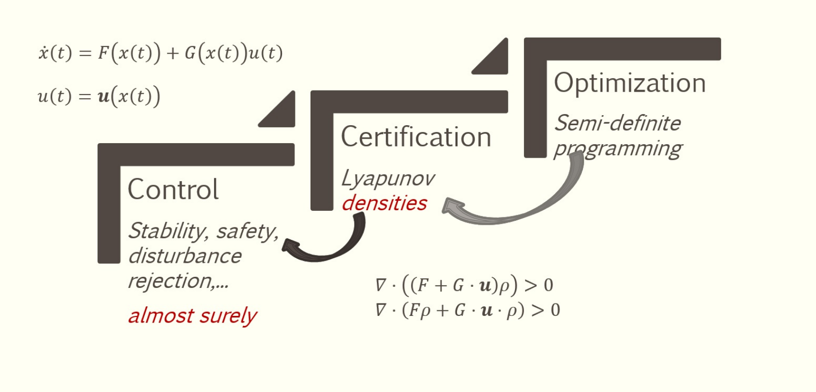


Certification of Global Attractors



Lyapunov, A. M. *A general task about the stability of motion*. Ph. D. Thesis, 1892.

Certification of Milnor Attractors



Rantzer, A. *A dual to Lyapunov's stability theorem*. 2001.

Can We Certify Synchronization of Oscillators For Almost All Initial Conditions?

Kuramoto Model

$$\dot{\theta}_i = \omega + \sum_{j=1}^{N} g(\theta_j - \theta_i), \qquad g(\varphi) = \sum_{k=1}^{L} \alpha_k \sin(k\varphi + \beta_k),$$

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$$\dot{\varphi}_k = \mathcal{F}_k(\varphi_1, \ldots, \varphi_{N-1}), \qquad \varphi_k = \theta_{t_h} - \theta_{t_k}$$

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$$\boxed{\text{Almost Global Synchronization of Kuramoto Model}}$$
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$$\boxed{\text{Origin of the Phase Difference Model is a Milnor attractor}}$$

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Phase Difference Model

$\dot{\varphi} = \mathcal{F}(\varphi), \qquad \varphi \in [0, 2\pi)^{N-1}.$

Using the stereographic projection $\varphi_k \to x_k = \cot(\varphi_k/2)$

Rational Function Model

$\dot{x} = F(x), \qquad x \in \mathbb{R}^{N-1}.$

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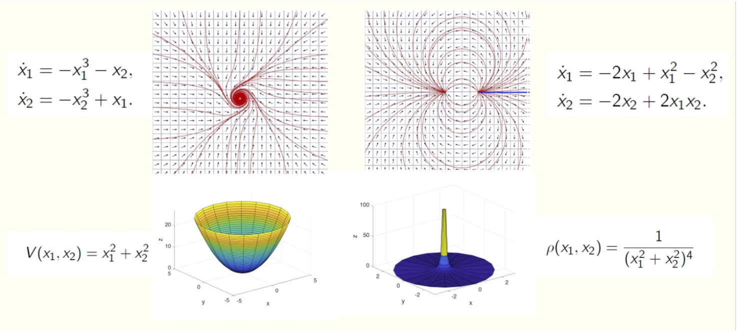
$$\Downarrow$$
$$\boxed{\text{The divergence of almost all solution on Euclidean space to infinity}}$$

Lyapunov Functions vs Lyapunov Densities

If $V(x)$ is nonnegative, radially unbounded and $\nabla V \cdot F < 0$ then $x(t) \to 0$.

If $\rho(x)$ is positive, integrable away from zero and $\nabla(\rho \cdot F) > 0$ then a.e. $x(t) \to 0$.



Divergence of Solutions to Infinity

Theorem

$$\boxed{\exists \text{ a positive } \rho \in C^1(\mathbb{R}^{N-1}, \mathbb{R}),\ \nabla \cdot (\rho F)(x) > 0 \text{ for } x \in \mathbb{R}^{N-1}.}$$

$$\Downarrow$$

$$\boxed{\text{Almost all solutions of } \dot{x} = F(x) \text{ diverge to infinity.}}$$

Rational Function Model: $F(x) = \dfrac{P(x)}{q(x)}$

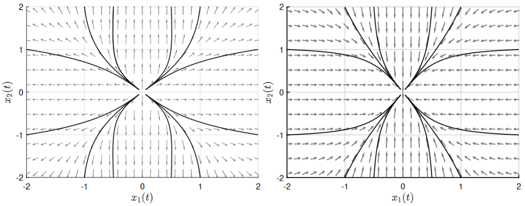
SOS Condition: $\nabla \cdot \left(\dfrac{P\rho}{q} \right) = \dfrac{\nabla \cdot (P\rho)q - P\rho \cdot \nabla q}{q^2} > 0$

Divergence of All Solutions to Infinity

Example

$$\dot{x}_1 = \frac{a x_1 x_2^2 + b x_1^3}{x_1^4 + x_2^4 + 1} \qquad \dot{x}_2 = \frac{c x_1^2 x_2 + d x_2^3}{x_1^4 + x_2^4 + 1}$$

Choosing $\rho(x_1, x_2) = x_1^4 + x_2^4 + 1$, we obtain $\nabla \cdot (F\rho) = (3b + c)x_1^2 + (a + 3d)x_2^2$.



(a) $a = -0.1$ $b = 0.8$ $c = 0.2$ and $d = 0.5$.

(b) $a = -0.01$ $b = -0.8$ $c = -0.01$ and $d = -0.3$.

Almost all solutions diverge to infinity if $3b + c > 0, \qquad a + 3d > 0$

From Rational Polynomial Model to Phase Difference Model

$$\boxed{\text{The divergence of almost all solutions on Euclidean space to infinity}}$$
$$\Downarrow \; ???$$
$$\boxed{\text{Origin of the Phase Difference Model is a Milnor attractor}}$$
$$\Updownarrow$$
$$\boxed{\text{Almost Global Synchronization of Kuramoto Model}}$$

From Rational Polynomial Model to Phase Difference Model

The divergence of almost all solution on Euclidean space to infinity

$\Downarrow$

$x^{\mathcal{T}}(t) = \left(x_1^{\mathcal{T}}(t), \ldots, x_{N-1}^{\mathcal{T}}(t)\right)$, $\|x^{\mathcal{T}}(t)\| \to \infty$, namely, $x_k(t) \to \infty$ for some k.

$\Downarrow$

Almost all sol. of phase dif. mod. converge to the set $\bigcup_{k=1}^{N-1} \{\varphi_k^{\mathcal{T}} = 0\}$.

$\Downarrow$

Almost all sol. of Kuramoto model converge to the set $\bigcup_{k=1}^{N-1} \left\{\theta_{t_k^{\mathcal{T}}} = \theta_{h_k^{\mathcal{T}}}\right\}$.

From Rational Polynomial Model to Phase Difference Model

Using the full permutation symmetry of Kuramoto model,

$$\boxed{\text{Almost all sol. of Kuramoto model converge to the set } \bigcup_{k=1}^{N-1} \left\{ \theta_{t_k^{\mathcal{T}}} = \theta_{h_k^{\mathcal{T}}} \right\}.}$$

$$\Downarrow$$

$$\boxed{\text{Almost all sol. of Kuramoto model converge to } \mathcal{W}(\mathcal{T}) = \bigcap_{\gamma \in \mathbb{S}^N} \bigcup_{k=1}^{N-1} \left\{ \theta_{\gamma(t_k^{\mathcal{T}})} = \theta_{\gamma(h_k^{\mathcal{T}})} \right\}.}$$

The Language of Equivariant Dynamics

$$\boxed{\text{Almost all sol. of Kuramoto model converge to } \mathcal{A}(\mathcal{T}) = \bigcap_{\gamma\in\mathbb{S}^N} \bigcup_{k=1}^{N-1} \left\{\theta_{\gamma(t_k^{\mathcal{T}})} = \theta_{\gamma(h_k^{\mathcal{T}})}\right\}}$$

Kuramoto model (F) is $\mathbb{S}_N$-symmetric: $F \circ \gamma = \gamma \circ F, \quad \forall \gamma \in \mathbb{S}_N$

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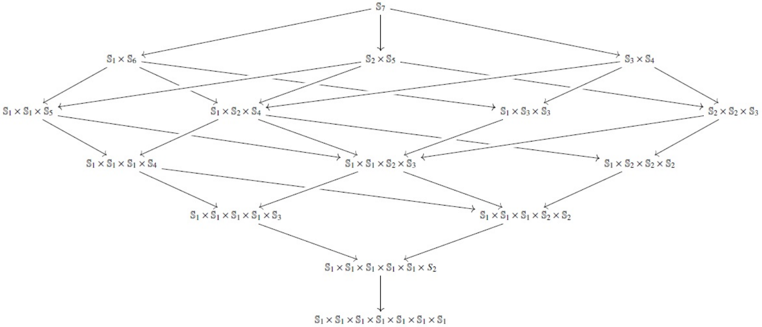
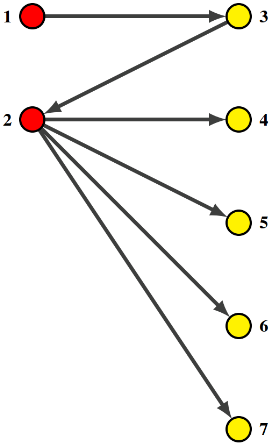
Γ_1 and Γ_2 are said to be conjugate if there exist a $\gamma \in \mathbb{S}_N$ such that $\Gamma_2 = \gamma^{-1}\Gamma_1 \circ \gamma$

Conjugacy Classes of Isotropy groups of S_7



Proper Colorings



Main Result

$\mathcal{I}_N$: Conjugacy classes of isotropy subgroups of $\mathcal{S}_N$
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$$\mathcal{A}(\mathcal{T}) := \bigcap_{\gamma \in \mathbb{S}^N} \bigcup_{k=1}^{N-1} \left\{ \theta_{\gamma(t_k^{\mathcal{T}})} = \theta_{\gamma(h_k^{\mathcal{T}})} \right\} = \bigcup_{\Gamma \in \mathbb{I}_N \setminus \mathbb{I}_{\mathcal{T}}} \bigcup_{\hat{\Gamma} \in \Gamma} \mathrm{Fix}(\hat{\Gamma}).$$

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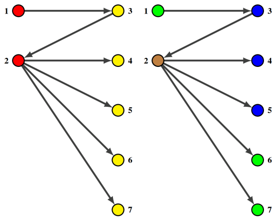
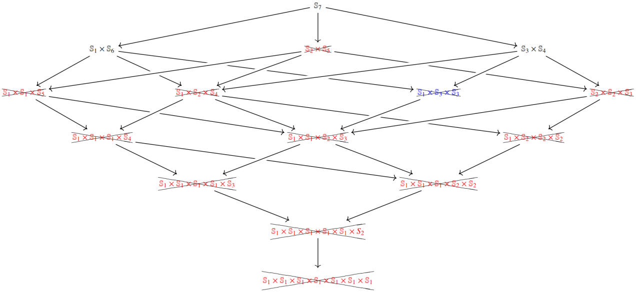
Theorem

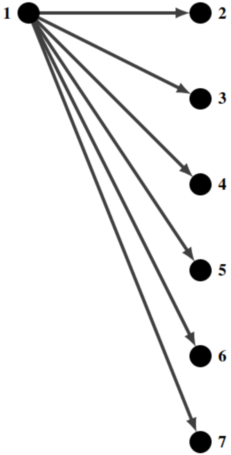
$$\boxed{\exists \text{ a positive } \rho \in C^1(\mathbb{R}^{N-1}, \mathbb{R}),\ \nabla \cdot (\rho F_\mathcal{T})(x) > 0 \text{ for almost all } x \in \mathbb{R}^{N-1}.}$$
$$\Downarrow$$
$$\boxed{\text{Almost all solutions of the Kuramoto model converge to } \mathcal{A}(\mathcal{T}).}$$

Colorings-Isotropy Classes

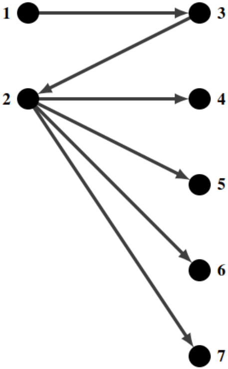




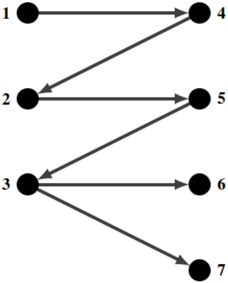
Canonical Trees For $N = 7$



(a) (1,6)-Tree

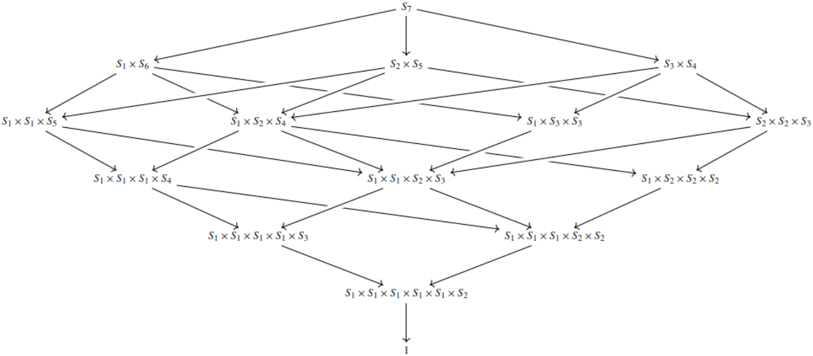


(b) (2,5)-Tree



(c) (3,4)-Tree

Isotropy group S_7



From Rational Polynomial Model to Phase Difference Model for Spanning Trees

Corollary

Let $\mathcal{T}_1, \ldots, \mathcal{T}_{\lfloor N/2 \rfloor}$ be spanning trees corresponding to partitions of N into $(1, N-1), \ldots, (\lfloor N/2 \rfloor, N - \lfloor N/2 \rfloor)$.

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Let $\mathcal{T}_1, \ldots, \mathcal{T}_{\lfloor N/2 \rfloor}$ be spanning trees corresponding to partitions of N into $(1, N-1), \ldots, (\lfloor N/2 \rfloor, N - \lfloor N/2 \rfloor)$.

Assume that $\forall k \in \{1, \ldots, \lfloor N/2 \rfloor\}, \exists$ a positive $\rho_k \in C^1(\mathbb{R}^{N-1}, \mathbb{R})$ such that $\nabla \cdot (\rho_k F_{\mathcal{T}_k})(x) > 0$ for $x \in \mathbb{R}^{N-1}$.

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Then, almost all solutions of the Kuramoto model are phase synchronized.

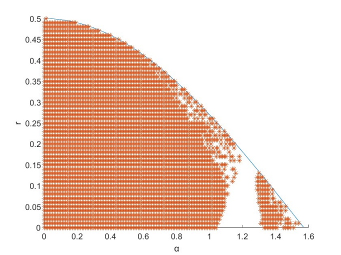
Almost global stability for Kuramoto Model for $N = 3$

Example

Consider the Kuramoto model for $N = 3$ (Ashwin, et.al., 2008) with the coupling function

$$g(x) = -\sin(x + \alpha) + r\sin(2x).$$

We define $\varphi_1 = \theta_1 - \theta_3$ and $\varphi_2 = \theta_2 - \theta_3$.



Almost Global Stability for Kuramoto Model for $N = 4$

Example

Consider the Kuramoto system for $N = 4$ (Ashwin, et.al., 2008) with the coupling function

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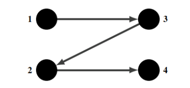
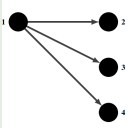
We choose a spanning tree $\mathcal{T}_1^4$ such that $\varphi_1 = \theta_1 - \theta_3, \varphi_2 = \theta_3 - \theta_2, \varphi_3 = \theta_2 - \theta_4$:

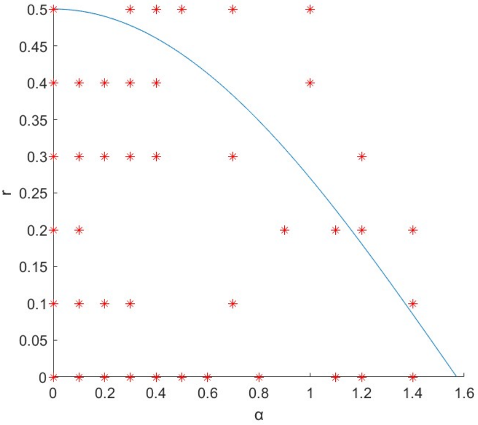


Now, we choose another three $\mathcal{T}_2^4$ such that $\varphi_1 = \theta_1 - \theta_2, \varphi_2 = \theta_1 - \theta_3, \varphi_3 = \theta_1 - \theta_4$:



Almost Global Stability for Kuramoto Model for $N = 4$

Parameter plane for $\mathcal{T}_1^4$



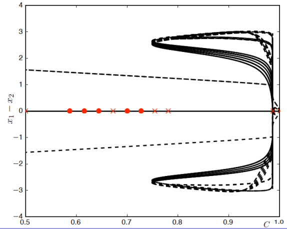
Van der Pol Oscillators

Example

Consider three identical van der Pol (VDP) systems given by

$$\epsilon \dot{x}_i = z_i + x_i - \frac{x_i^3}{3} + \sum_{j \neq i} \alpha(x_j - x_i),$$
$$\dot{z}_i = (c - x_i), \quad i = 1, 2, 3.$$

For $0.5 < c < 1$, the bifurcation diagram of c is



Van der Pol Oscillators

For the coupling function

$$g(x) = \alpha_1 \sin(x + \beta_1) + \alpha_2 \sin(2x + \beta_2) + \alpha_3 \sin(3x + \beta_3),$$

c	ρ
0.5	NONE
0.5867	$3.08 - 1.12y_1 - 1.12y_2 + 2.52y_1^2 - 0.2y_1y_2 + 2.52y_2^2$
0.6153	$3.09 - 2.05y_1 - 2.05y_2 + 2.56y_1^2 + 2.44y_1y_2 + 2.56y_2^2$
0.6437	$2.67 - 3.05y_1 - 3.05y_2 + 2.55y_1^2 + 1.88y_1y_2 + 2.55y_2^2$
0.6718	NONE
0.6996	$2.16 - 3.56y_1 - 3.56y_2 + 2.54y_1^2 + 3.91y_1y_2 + 2.54y_2^2$
0.7269	$1.98 - 3.66y_1 - 3.66y_2 + 2.61y_1^2 + 4.5y_1y_2 + 2.61y_2^2$
0.7539	NONE
0.7802	NONE
0.986	NONE
0.9998	NONE

Future Works

- Other connection structures
- Designing synchronizing feedback