GÖDEL AND TURING ON MATHEMATICAL INTUITION: BETWEEN PLATONISM AND EMBODIMENT

Francesco Beccuti

Università di Cagliari

francesco.beccuti@unica.it

ABSTRACT

This paper explores and elaborates on the views on mathematical intuition expressed in the writings of Kurt Gödel and Alan Turing as connected to the incompleteness phenomenon. While Gödel famously posited intuition as a suprasensible faculty directed toward abstract entities, Turing's considerations go in the direction of understanding it as intimately connected to sensory perception. As we will see, Turing's reflections, while unsystematized, offer an important point of dialogue and comparison to materialist, human-centered, and embodied approaches to mathematics and mathematics education.

The faculty or experience of mathematical intuition has traditionally been contrasted with the relatively straightforward recognizability of formal mathematical reasoning. Indeed, to an extent, a large part of the mathematical endeavour since Euclid consists in extending formalization in order to eliminate or diminish to the minimum appeals to intuition (Fischbein, 2002, p. 8), usually described as the faculty which constitutes our way of accessing mathematical truths directly or immediately.

Both Alan Turing and Kurt Gödel, reflecting on the incompleteness phenomenon of mathematical logic, similarly suggested the impossibility or unfeasibility of formally circumventing the role of intuition in mathematical practice. However, different philosophical accounts of the nature of intuition can be extracted from their reflections. On the one hand, Gödel understood intuition as a faculty akin to sensory perception but categorically distinct from it, in accordance with his own Platonist philosophy of mathematics. On the other hand, the later reflections of Turing, while unsystematized, seemingly go in the direction of characterizing intuition as inextricably linked to or derived from sensory perception.

In this paper I will first introduce Gödel's account of intuition contrasted with the syntactic view of mathematics advanced by the main protagonists of the early Vienna school. I will then present Turing's early account of intuition framed within his reflections on the mechanizability of mathematical reasoning which I will explain via three simple mathematical statements. Moreover, I will expose Gödel's view on the non-sensible nature of intuition in the context of his own Platonism, which he connected to the Kantian tradition and Husserlian phenomenology. Finally, I will describe Turing's later case for the development of cognitive (embodied) mechanical models of the evolving mathematical mind as a concrete proposal for carrying out his project of endowing machines with general reasoning capabilities. Throughout the paper, I will also highlight notable similarities between the introspective and theoretical considerations articulated by Turing and Gödel and the earlier comparable reflections of Henri Poincaré on the epistemology of mathematics, as well as the analogous conclusions later discussed on the basis of experimental evidence by psychologists and cognitive scientists.

I will conclude by arguing that Turing's reflections, while not being decisive in establishing the sensible nature of intuition, bear significantly on the debate on the sensibility of mathematical intuition and the connected perception of mathematics' "purity" within the philosophical and the educational discourse, emerging as promisingly relevant to materialistic and human-centered philosophies of mathematics and mathematics education.

1. Gödel and the syntactic view

As said, an important part of the mathematical endeavour since Euclid, at least, consists in delimiting or eliminating intuition in favour of formalization (Fischbein, 2002). A radical position in this respect is what Gödel referred to as the *syntactic view* of mathematics, an extreme version of formalism, i.e., the philosophical doctrine affirming that mathematics is just a game of syntax with no actual content.[1] Among the champions of this approach were, for instance, Rudolf Carnap and Hans Hahn, among the founders and main protagonists of the Vienna Circle.[2]

Gödel's unpublished paper *Is mathematics syntax of language?*, dating around 1953, contains an attack against the syntactic view, a stance which Gödel later attributed to the spirit of the

[1] However, according to Detlefsen (2005), the term "formalism" would encompass a broader spectrum of doctrines. I will not further elaborate on this issue here.

[2] Presumably partially under the early influence of Ludwig Wittgenstein's *Tractatus Logico-Philosophicus.* See Wang (1996, pp. 177–182) for Gödel's (critical) comments on Wittgenstein's philosophy. Hahn's and Carnap's view was strongly motivated by a rejection and overall distrust of intuition in mathematics. In particular Hahn viewed the history of mathematics as a progressive departure from the intuitive towards the formal, which, he foresaw, will eventually see the complete elimination of the former in favour of the latter. "Because intuition turned out to be deceptive in so many instances, and because propositions that had been accounted true by intuition were repeatedly proved false by logic, mathematicians became more and more sceptical of the validity of intuition. They learned that it is unsafe to accept any mathematical proposition, much less to base any mathematical discipline on intuitive convictions. Thus a demand arose for the expulsion of intuition from mathematical reasoning, and for the complete formalization of mathematics" (Hahn, 1933, p. 93). For a historical perspective see Kline (1982, Chapter X and XI).

science of his times which he condemned as reductionist, materialistic, and overall too much prejudiced against metaphysics.[3] He summarized the syntactic view as follows.

> Mathematics can be completely reduced to (and in fact *is* nothing but) syntax of language. I.e., the validity of mathematical theorems consists solely in their being consequences of certain syntactical conventions about the use of symbols, not in their describing states of affairs in some realm of things. Or, as Carnap puts it: *Mathematics is a system of auxiliary sentences without content or object.*[4] […] The idea at the bottom of this interpretation of logical truth is the following: The meaning of sentences is defined by […] rules which determine under which circumstances a given sentence can be asserted. These rules, in certain limiting cases, may have the consequences that a sentence can be asserted under *all* circumstances (e.g., it will rain or it will not rain). This makes such sentences true but void of content. Of course rules especially referring to these limiting cases may then be formulated, which is done in the axioms and rules of pure mathematics. Such rules may be called "syntactical" because they do not refer to meaning and, therefore, assertions in conflict with them are excluded already because of their structure, exactly as assertions which do not conform to the rules of grammar. In his *Logical Syntax and Language* […] Carnap has carried this program out. […] Finally much of the work of the Hilbert school about the formalization and consistency of mathematics can be interpreted to be a partial elaboration of this view, although the authors of these papers, for the most part, favour different philosophical opinions. (Gödel, 1953, pp. 335–336)

Gödel deemed the syntactic conception "refutable, as far as any philosophical assertion can be refutable in the present state of philosophy" (Goldfarb, 1995, p. 324).[5] The kernel of Gödel's criticism is based on the fact that regarding mathematics as solely the act of combining formal rules, essentially would make the activity of the mathematicians reducible to that of an arranger of finite chains of inferences completely void of content. However, in this perspective, the only legitimate reason for choosing as a start of a mathematical reasoning one specific system of premises (among the infinite possible choices one could make) would be that the chain of

---

[3] See Wang (1996, pp. 172–177) for Gödel's (very critical) views on positivism. See also Gödel's sketch of his own history of ideas in Wang (1996, p. 156) as well as the many scattered remarks in his *Nachlass* regarding what he calls "the spirit of the times" which he perceived to be countering. Cf. incidentally Hahn (1930): "If it seems to a superficial observer that the scientific world view we have sketched stands opposed to the spirit of the times – which at present tends towards metaphysics, towards connections accessible only to mystical intuition and apprehensible only by feeling, and which aims at the whole, avoiding minutely detailed work" (p. 30).

[4] Here a note in Gödel's text says: "I would like to say right here that Carnap today would hardly uphold the formulations I have quoted […]". In other words, Carnap withdrew some of his early formalistic attitude, possibly because of Gödel's very influence. See Wang (1996, pp. 174–177) for the later Gödel's very harsh opinion on positivists and "those who, like Carnap, misuse symbolic language want to discredit mathematical logic; they want to prevent the appearance of philosophy. The whole movement of positivists wants to destroy philosophy".

[5] Cf. Gödel (1951): "The most I could assert would be to have disproved the nominalistic view, which considers mathematics to consist solely in syntactical conventions and their consequences" (p. 322).

inferences which could be derived from such premises does not end with a contradiction (i.e., the only legitimate reason for choosing one system of premises over another would be its consistency). But, as argued by Kennedy (2020b),

> consistency will never be intrinsic to the system; it must always be imported "from the outside," so to speak, as follows from the Second Incompleteness Theorem, which states that consistency is not provable from within any system adequate to formalize mathematics. (p. 1)

This is the reason why, for Gödel, mathematical propositions cannot be void of content or based on arbitrary stipulations.[6] In particular,

> the *abstract* mathematical concepts, such as "infinite set," "function," etc., cannot be proved consistent without again using abstract concepts, i.e., such as are not merely ascertainable properties or relations of finite combinations of symbols. So, while it was the primary purpose of the syntactical conception to justify the use of these problematic concepts by interpreting them syntactically, it turns out that quite on the contrary, abstract concepts are necessary in order to justify the syntactical rules […] (p. 357, emphasis in the original)

It follows that, according to Gödel, any attempt to reduce mathematics to a set of syntactic rules would be doomed to failure.

> More precisely the situation can be described as follows: That mathematics *does* have a content (in any acceptable sense of the term) appears from the fact that, in whatever way it, or any part of it, is built up, one always needs certain undefined terms and certain axioms (i.e., deductively unprovable assertions) about them. *For these axioms there exists no other rational (and not merely practical) foundation except either that they (or propositions implying them) can directly perceived to be true (owing to the meaning of the terms or by an intuition of the objects falling under them), or that they are assumed (like physical hypotheses) on the grounds of inductive arguments, e.g. their success in the applications* […] *To eliminate mathematical intuition or empirical induction by positing the mathematical axioms to be true by convention is not possible.* (Gödel, 1953, pp. 346–347, emphasis in the original)

Therefore, for Gödel, mathematics is in need of some ultimate content acting as a foundation and this content is (grasped by) intuition: an immediate form of perception of mathematical concepts and facts which appears to be impossible to eliminate in view of the incompleteness phenomenon.

[6] Cf. Ernest (2023): "a rule of itself, in a vacuum, cannot determine a unique direction in which the application takes us. Mathematics is not made up of purely mechanical rule following" (p. 24).

2. Turing's distinction

Alan Turing's 1936 paper *On computable numbers, with an application to the Entscheidungsproblem* (Turing, 1937) marked the start of his investigations on the possibility and the limits of mechanizing the mathematical endeavor. Famously, he introduced the concept of the discrete-state (now-called Turing) machine which would later turn to be pivotal in the development of theoretical computer science. Via this concept he tried to argue for the possibility of reducing the human mathematical activity to finite, tangible, and machine-executable procedures. As Cockshott (2012) argued,

> Turing's first proposal for a universal computing machine is based on an implicit rejection of this view [Platonism]. His machine is intended to model what a human mathematician does when calculating or reasoning, and by showing what limits this machine encounters, he identifies constraints which bind mathematical reasoning in general (whether done by humans or machines). From the beginning, he emphasises the limited scope of our mental abilities and our dependence on artificial aids — pencil and paper for example — to handle large problems. We have, he asserted, only a finite number of 'states of mind' that we can be in when doing calculation. We have in our memories a certain stock of what he calls 'rules of thumb' that can be applied to a problem. Our vision only allows us to see a limited number of mathematical symbols at a time and we can only write down one symbol of a growing formula or growing number at a time. The emphasis here, even when he looks at the human mathematician, is on the mundane, the material, the constraining (p. 1; see also Hodges, 2012).[7]

However, Turing's attempt was not entirely conclusive, as we will see. Indeed, in his later 1939 doctoral thesis, Turing, reflecting again on mathematical reasoning in general, singled out two distinguishable faculties which characterize it: intuition and ingenuity.

> Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two faculties, which we may call *intuition* and *ingenuity*. The activity of the *intuition consists in making spontaneous judgments which are not the result of conscious trains of reasoning* […] The exercise of *ingenuity* in mathematics *consists in aiding the intuition through suitable arrangements of propositions, and perhaps geometrical figures or drawings*. It is intended that when these are really well arranged the validity of the intuitive steps which are required cannot seriously be doubted. The parts played by these two faculties differ of course from occasion to occasion, and from mathematician to mathematician. This arbitrariness can be removed by the introduction of a formal logic. The necessity for using the intuition is then greatly reduced by setting down formal rules for carrying out inferences which are always intuitively valid. When working with a

[7] See also Cockshott et al. (2012). The faith in the possibility of reducing the mental to the mechanical foregrounded what Gödel deemed Turing's main "philosophical error" (1972, p. 306; cf. Hodges, 1998). However, against this interpretation of Turing's view see Piccinini (2003, p. 25).

> formal logic, the idea of ingenuity takes a more definite shape. In general a formal logic, will be framed so as to admit a considerable variety of possible steps in any stage in a proof. Ingenuity will then determine which steps are the more profitable for the purpose of proving a particular proposition. (Turing, 1939, pp. 214–215, emphasis added)

Thus, for Turing, intuition is the faculty allowing us to primitively sense the truth of mathematical statements without further argument. On the other hand, ingenuity is the faculty that helps in rigorously deducing mathematical statements from a set of premises. Notice incidentally that Turing's distinction appears to resonate with the more recent dual-process accounts of human thinking formulated within cognitive psychology and popularized in Daniel Kahneman's 2011 book *Thinking fast and slow*. These theories categorize thinking into two kinds: one unconscious, fast, and intuitive which has similarities with perception; the other conscious, slow, and reflective overseeing hypothetical thinking and deduction. See Evans (2012) and Evans & Stanovich (2013) together with the associated open commentary on *Perspectives on Psychological Science* for an account of the mainstream version of this theory and for some relatively recent critiques.[8]

But what exactly had Turing in mind? After distinguishing between intuition and ingenuity, he wrote that this distinction takes a more definite and objective form when a mathematical argument, being restated in terms of formal logic, takes a precise shape as (finite) sequence of sentences leading from the premises of the argument to its conclusion. I will illustrate this by analysing three relatively simple mathematical statements.

> *Example 1.* Seven plus five is twelve:[9]
>
> $$7 + 5 = 12.$$
>
> A statement like this is seldom proved and most people just can recall it from memory as a consequence of extended exercises in primary school. However, if one

[8] Dual-process theories of reasoning have been criticized by many on various grounds (see, e.g. Gigerenzen, 2011; Osman, 2013). Typically, critics have pointed out that the distinction is quantitative rather than qualitative and that it is not obvious that the result of the aforementioned experiments tells us where exactly to place the boundary between intuition and reasoning.

[9] This is the famous Kantian statement which Kant discussed in his *Critique of pure reason* (A7/B11, B15–16, A13/B26, 1781), concluding that it is a synthetic a priori judgement, because it is non-empirical, necessarily true and informative (for the concept of seven and five are not contained in the concept of twelve). Schopenhauer, in his *On the fourfold root of the principle of sufficient reason,* substantially agreed with Kant and noticed that $7 + 5 = 12$ is not really an identity statement as $12 = 12$ (Schopenhauer, 1813, p. 200). Furthermore, in *The world as will and representation* he remarks, regarding arithmetic, that "an intuitive or perceptive evidence of every sum or calculation is made possible". He further argued that, more generally, "there will be no doubt that the evidence of mathematics, which has become the pattern and symbol of all evidence, rests essentially not on proofs, but on immediate intuition or perception. Here, as everywhere, that is the ultimate ground and source of all truth" (Schopenhauer, 1819, pp. 75–76).

perhaps is unsure, the simplest method to find the result of $7 + 5$ is perhaps the following: starting from 7, one counts up five times (i.e. adds 1 to 7 five times), so that $7 + 5 = 8 + 4 = 9 + 3 = 10 + 2 = 11 + 1 = 12$.

Example 2. If a number is smaller than another number, then the square of the former is smaller than the square of the latter:

$$p < q \rightarrow p^2 < q^2, \text{ for } p, q \in \mathbb{N}.$$

In ordinary mathematical language, we could say that a proof of this statement is elementarily obtained by the application of an axiomatic definition of "being smaller than": p is smaller than q means that there is some natural number c such that $p + c = q$. Therefore, we can write $(p + c)^2 = q^2$ and hence $p^2 + c^2 + 2pc = q^2$, by elementary arithmetical properties. Now, by positing $r = c^2 + 2pc$, we have $p^2 + r = q^2$ and hence $p^2 < q^2$, again by the definition of "being smaller than".

Example 3. The sum of the first k numbers is the k^{th} triangular number:

$$1 + \cdots + k = \frac{k(k+1)}{2}, \text{ for } k \in \mathbb{N}.$$

Here we may use Peano's inductive axiom of arithmetic: *if* a property is true for 0 and, assuming it true for k, it is also true for $k + 1$, *then* the property is true for all ks. Clearly $0 = \frac{0(0+1)}{2}$. If we assume that $1 + \cdots + k = \frac{k(k+1)}{2}$, then we can conclude $1 + \cdots + k + (k + 1) = \frac{k(k+1)}{2} + (k + 1) = \frac{(k+1)(k+2)}{2}$, by elementary calculations.

Now, all the somewhat informal proofs given above can be transformed into pedantic step-by-step formal proofs by application of the usual logical rules of inference. For instance, in *Figure 1*, it is shown how one can derive $7 + 5 = 12$ by repeated application of the theorem

$$\forall x, y(x + (y + 1) = (x + 1) + y),$$ [10]

in a simplified version of Gentzen's sequent calculus. Similarly, in *Figure 2*, the definition of "being smaller than" is expressed axiomatically by the statement

$$\forall x, y(x < y \leftrightarrow \exists z(z > 0 \land x + z = y))$$

and the proof is given by careful and subsequent use of other axioms and simple theorems, overall resembling the informal proof given in *Example 2*. Finally, *Figure 3* displays the formal proof of the statement in Example 3.

[10] This in turn may be derived from the associativity law (usually taken as an axiom) and from the commutativity law (which may be taken as an axiom or derived from Peano's induction axiom). The same holds for the other examples (see Note 11).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{Theorem}\ \forall x, y\ \xi(x,y)}{\forall y\ \xi(7,y)}}{7+5=8+4} \quad \dfrac{\dfrac{\textbf{Theorem}\ \forall x, y\ \xi(x,y)}{\forall y\ \xi(8,y)}}{8+4=9+3}}{7+5=9+3} \quad \dfrac{\dfrac{\textbf{Theorem}\ \forall x, y\ \xi(x,y)}{\forall y\ \xi(9,y)}}{9+3=10+2}}{7+5=10+2} \quad \dfrac{\dfrac{\textbf{Theorem}\ \forall x, y\ \xi(x,y)}{\forall y\ \xi(10,y)}}{10+2=11+1}}{7+5=11+1} \quad \dfrac{\dfrac{\textbf{Theorem}\ \forall x, y\ \xi(x,y)}{\forall y\ \xi(11,y)}}{11+1=12+0}}{7+5=12+0} \quad \dfrac{\textbf{Axiom}\ \forall x\ (x+0=x)}{12+0=12}}{7+5=12}}{}$$

Figure 1. Proof of $7 + 5 = 12$, where $\xi(x, y)$ is $x + (y + 1) = (x + y) + 1$

Generally speaking, as in these examples, the proof of any mathematical statement can be arranged as a formal proof which has a structure similar to that of a tree, where the leaves are the premises of the argument and the root is the conclusion of the argument. Starting from the bottom of the tree, the first node (the root of the tree) contains the sentence we want to prove, while each line above follows directly from one or more premises by application of a rule of logic. At the very top of the tree (at the leaves) are those sentences which can be taken to be true without proof: either axioms or previously established theorem.[11]

In this way the appeal to intuition is confined to the strictly necessary while ingenuity takes a very definite form, to the extreme extent that it would become completely unnecessary (at least in theory) to the activity of the mathematician:

> We are always able to obtain from the rules of a formal logic a method of enumerating the propositions proved by its means. We then imagine that all proofs take the form of a search through this enumeration for the theorem for which a proof is desired. In this way ingenuity is replaced by patience. (Turing, 1939, p. 215)

Indeed, since each rule of logic can use only a finite number of premises, a machine (or a human with plenty of time and patience), can just enumerate all the possible premises of a statement and, starting from the bottom, build all the possible trees having such statement as root. It follows that we can program a machine which, having as input a statement, a chosen set of axioms and a chosen set of rules, will subsequently build all the possible trees having as root the chosen statement and such that each node is obtained by application of one of the rules. Furthermore, we can ask the machine to terminate its computation when a tree having only axioms as leaves is found.

[11] Of course, the latter are in turn the roots of their own proof trees which will have as leaves either axioms or previously established theorems. By "unfolding" every theorem into a tree and by "gluing" all the related trees together, eventually we can represent the proof of any theorem as a tree having only axioms as leaves.

Axiom

$$\forall x, y(x < y \leftrightarrow \exists z(x + z = y))$$
$$\forall y(a < y \leftrightarrow \exists z(a + z = y))$$
$$a < b \leftrightarrow \exists z(a + z = b)$$
$$a < b \rightarrow \exists z(a + z = b)$$
$$\exists z(a < b \rightarrow (a + z = b))$$
$$a < b \rightarrow a + c = b$$

Theorem

$$\forall x, y(x = y \rightarrow x^2 = y^2)$$
$$\forall y(a + c = y \rightarrow (a + c)^2 = y^2)$$
$$a + c = b \rightarrow (a + c)^2 = b^2$$

$$a < b \rightarrow (a + c)^2 = b^2$$

Theorem

$$\forall x, y(x + y)^2 = x^2 + y^2 + 2xy$$
$$\forall y(a + y)^2 = a^2 + y^2 + 2ay$$
$$(a + c)^2 = a^2 + c^2 + 2ac$$

$$a < b \rightarrow (a^2 + c^2 + 2ac = b^2)$$
$$\exists z(a < b \rightarrow (a^2 + z = b^2))$$
$$a < b \rightarrow \exists z(a^2 + z = b^2)$$

Axiom

$$\forall x, y(x < y \leftrightarrow \exists z(x + z = y))$$
$$\forall y(a^2 < y \leftrightarrow \exists z(a^2 + z = y))$$
$$a^2 < b^2 \leftrightarrow \exists z(a^2 + z = b^2)$$
$$\exists z(a^2 + z = b^2) \leftrightarrow a^2 < b^2$$
$$\exists z(a^2 + z = b^2) \rightarrow a^2 < b^2$$

$$a < b \rightarrow a^2 < b^2$$
$$\forall q(a < q \rightarrow a^2 < q^2)$$
$$\forall p, q(p < q \rightarrow p^2 < q^2)$$

Figure 2. Proof of $\forall p, q \in \mathbb{N}\ (p < q \rightarrow p^2 < q^2)$

Theorem

$$0 = \frac{0(0+1)}{2}$$

Axiom

$$\forall x, y, z(x = y \rightarrow x + z = y + z)$$
$$\forall y, z(1 + \cdots + n = y \rightarrow 1 + \cdots + n + z = y + z)$$
$$\forall z(1 + \cdots + n = \tfrac{n(n+1)}{2} \rightarrow 1 + \cdots + n + z = \tfrac{n(n+1)}{2} + z)$$
$$1 + \cdots + n = \tfrac{n(n+1)}{2} \rightarrow 1 + \cdots + n + n + 1 = \tfrac{n(n+1)}{2} + (n + 1)$$

Theorem

$$\forall x, y, z(\tfrac{x}{y} + z = \tfrac{x+yz}{y})$$
$$\forall y, z(\tfrac{n(n+1)}{y} + z = \tfrac{n(n+1)+yz}{y})$$
$$\forall z(\tfrac{n(n+1)}{2} + z = \tfrac{n(n+1)+2z}{2})$$
$$\tfrac{n(n+1)}{2} + (n + 1) = \tfrac{n(n+1)+2(n+1)}{2}$$

Theorem

$$\forall x, y, z(xy + zy = y(x + z))$$
$$\forall y, z(ny + zy = y(n + z))$$
$$\forall z(n(n + 1) + z(n + 1) = (n + 1)(n + z))$$
$$n(n + 1) + 2(n + 1) = (n + 1)(n + 2)$$

$$\tfrac{n(n+1)}{2} + (n + 1) = \tfrac{(n+1)(n+2)}{2}$$

$$1 + \cdots + n = \tfrac{n(n+1)}{2} \rightarrow 1 + \cdots + n + 1 = \tfrac{(n+1)(n+2)}{2}$$
$$\forall k(1 + \cdots + k = \tfrac{k(k+1)}{2} \rightarrow 1 + \cdots + k + 1 = \tfrac{(k+1)(k+2)}{2})$$

$$0 = \tfrac{0(0+1)}{2} \wedge \forall k(1 + \cdots + k = \tfrac{k(k+1)}{2} \rightarrow 1 + \ldots k + 1 = \tfrac{k+1}{k+2})$$

Axiom

for all P and x free in P

$$(P(0) \wedge \forall k(P(k) \rightarrow P(k + 1)) \rightarrow \forall k P(k)$$

$$\forall k(1 + \cdots + k = \tfrac{k(k+1)}{2})$$

Figure 3. Proof of $\forall k \in \mathbb{N}\ (1 + \cdots + k = \frac{k(k+1)}{2})$

3. The seemingly unavoidable role of intuition

Now, it could appear from overlooking the former discussion that it could be possible to reduce the mathematical activity as a whole to rule-following mechanical procedures. However, following Turing,

1) even if we can program a machine that, say, finds a proof-tree starting from the axiom that expresses the meaning of "being smaller than" down to the root $p < q \rightarrow p^2 < q^2$, such machine would not appear to be *understanding* what it *means* "to be smaller than". The machine was just hard-wired the axiom $\forall x, y\ (x < y \leftrightarrow \exists z(x + z = y))$ as a legal move in this abstract proof-tree-building game.[12] On the contrary, it appears that we are able to introspectively *see* or *perceive* the meaning of "being smaller than" based on something like a representation of numbers as being placed one after another on an imaginary line;[13]

2) we *knew in advance* that there was a proof of $p < q \rightarrow p^2 < q^2$. It follows that a suitably programmed machine would eventually find it for good. However, if we are presented with another statement (of which we do not know whether there is a proof or not), then a similarly programmed machine would eventually find a proof only if a proof of the statement indeed exists. On the contrary, if no such proof exists, the machine will go on building tree after tree indefinitely, never terminating its computation.

Similar remarks can be made concerning the formal proofs of $7 + 5 = 12$ and $1 + \cdots + k = \frac{k(k+1)}{2}$. Thus, in general, and in agreement with Gödel's argument of the previous section, intuition appears to remain an unavoidable component in mathematical practice for

1) perceiving the truth of axioms and the correctness of definitions;

2) perceiving that a proof of a statement exists if it does or does not exists if it does not.

[12] Cf. the following quotation of Poincaré on the difference between playing chess mechanically and understanding the game: "If you are present at a game of chess, it will not suffice, for the understanding of the game, to know the rules for moving the pieces. That will only enable you to recognize that each move has been made conformably to these rules, and this knowledge will truly have very little value. Yet this is what the reader of a book on mathematics would do if he were a logician only. To understand the game is wholly another matter; it is to know why the player moves this piece rather than that other which he could have moved without breaking the rules of the game. It is to perceive the inward reason which makes of this series of successive moves a sort of organized whole. This faculty is still more necessary for the player himself, that is, for the inventor" (Poincaré, 1907, p. 22).

[13] Clearly, we could define $x < y$ in an entirely different way, but in this case the symbol $<$ would not represent the concept of "being smaller than" anymore.

These objections to the possibility of mechanizing mathematics echo the remarks made by Henri Poincaré concerning the possibility of reducing mathematics to logic (meaning rule-following procedures):

> The logician cuts up, so to speak, each demonstration into a very great number of elementary operations; when we have examined these operations one after the other and ascertained that each is correct, are we to think we have grasped the real meaning of the demonstration? […] Pure analysis puts at our disposal a multitude of procedures whose infallibility it guarantees; it opens to us a thousand different ways on which we can embark in all confidence; we are assured of meeting there no obstacles; but of all these ways, which will lead us most promptly to our goal? Who shall tell us which to choose? We need a faculty which makes us see the end from afar, and intuition is this faculty: it is necessary to the explorer for choosing his route; it is not less so to the one following his trail who wants to know why he chose it. (Poincaré, 1907, pp. 21–22)

Nonetheless, contrary to Poincaré's opinion, it was by some believed in the past that it would eventually be feasible to indefinitely extend the mechanistic endeavour to the point of programming a machine capable of completely replacing humans in doing mathematics (this was roughly Hilbert's program, which suffered perhaps from a nominalistic or reductionistic prejudice, though Hilbert himself did not endorse nominalism as his own philosophy of mathematics, see Kennedy, 2020a; Zach, 2023). However, it turns out from Gödel's incompleteness theorems that it is not practically possible to eliminate intuition from mathematics.

> In pre-Gödel times it was thought by some that it would probably be possible to carry this programme to such a point that all the intuitive judgments of mathematics could be replaced by a finite number of these rules. The necessity for intuition would then be entirely eliminated. In our discussions, however, we have gone to the opposite extreme and eliminated not intuition but ingenuity, and this in spite of the fact that our aim has been much the same direction. (Turing, 1939, p. 215)

What stems from the incompleteness theorems is that there is no enumeration of all the possible axioms of mathematics that can be recognised as such.[14] Furthermore, Turing's theorem (itself a version of the second incompleteness theorem for theoretical computer science proved by Turing in his 1936 paper) ensures that it is not in general possible to build a (Turing) machine that decides in advance whether there is or there is not a proof of a given statement. Therefore, as said, even if it turns out to be possible to replace ingenuity with good programming, it is not possible to circumvent the role of intuition in mathematical practice.

---

[14] "For, it [the second incompleteness theorem] makes it impossible that someone should set up a certain well-defined system of axioms and rules and consistently make the following assertion about it: All of these axioms and rules I perceive (with mathematical certitude) to be correct, and moreover I believe that they contain all of mathematics. If someone makes such a statement he contradicts himself" (Gödel, 1951, p. 309; see also Beccuti, 2018).

Interestingly, cognitive psychologist Stanislas Dehaene has also defended the appeal to intuition in scientific and mathematical discourse on a similar basis in his 1997 book *The Number Sense* where he explicitly argued from the incompleteness theorems.

> Our best system of axioms fails to capture, in a unique way, our intuitions of what numbers are. […] Thus, our "number sense" cannot be reduced to the formal definition provided by these axioms. […] providing a univocal formal definition of what we call numbers is essentially impossible: The concept of number is primitive and undefinable. (Dehaene, 1996, p. 264)

Dehaene's remarks originate not only from the aforementioned theoretical consequences of the incompleteness theorems as well as from his good reads (e.g. Kant, Husserl, Poincaré), but also from his own seminal empirical studies on the cognitive psychology of the mathematical reasoning. These would show that humans (and, partially, animals) are provided at birth with an innate and evolutionary-determined "sense of number" which would enable them to count, compare, add, and subtract numbers. Furthermore, this capacity would be independent from language and connected to neuronal circuits located in the inferior parietal region of both cerebral hemispheres (Dehaene, 1997, pp. 244–245). Among other things, Dehaene is also praised for the mental number line account of number representation. According to this, numbers are represented by the mind on a horizontal line in ascending order from left to right or from right to left, a circumstance usually found to depend on the tested subjects' writing direction (Dehaene et al., 1993).

4. The sensible nature of intuition?

Not surprisingly, Dehaene's proposal fits very well with the standard introspective representation of numbers. Indeed, looking again at *Example 2*, as said above, one could easily claim that the truth of the definition of "being smaller than", $p < q \leftrightarrow \exists z(p + z = q)$, ultimately rests on the correctness of our overall mental picture of numbers as resting on a line separated by magnitudes (or distances) which themselves can be made to correspond in turn to numbers on the line. In this representation (see *Figure 4*)

- "being smaller than" means "being on the left with respect to";
- "adding c to p" means moving c steps on the line starting from p;
- "adding $c > 0$ to p" means moving c steps from p to the right, while "adding $c < 0$ to p" (i.e. subtracting $-c > 0$ from p) means moving $-c$ steps from p to the left;
- 0 is the unique magnitude that if added to another quantity corresponds to not moving;

It is straightforward to see that, within the spatial (or geometric) structure just defined, it is the case that $5 + 7 = 7 + 5 = 12$, simply because moving seven steps to the right starting from place 5 is the same as moving five steps to the right starting from place 7.

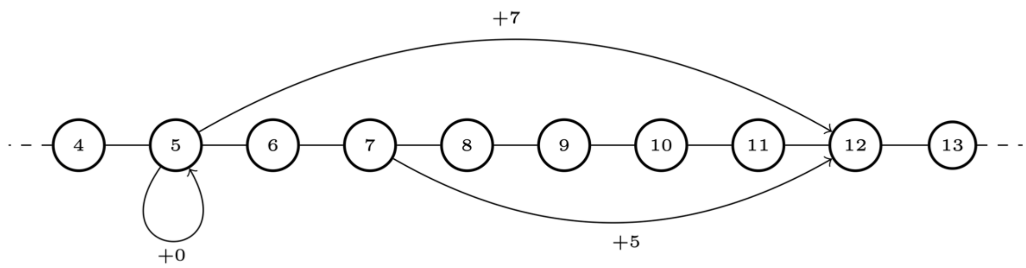

Figure 4. The number line

But why is it straightforward to "see" it in all possible instances? Why are we so easily ready to admit that this is the case for all numbers? Since the number line representation which *Figure 4* suggests is infinite (or infinitely extensible), it is not the case that we can try all the possible pairs of numbers (x, y) and convince ourselves that $x + y = y + x$ for all such pairs. And even if the structure in question were finite, we would never do that.

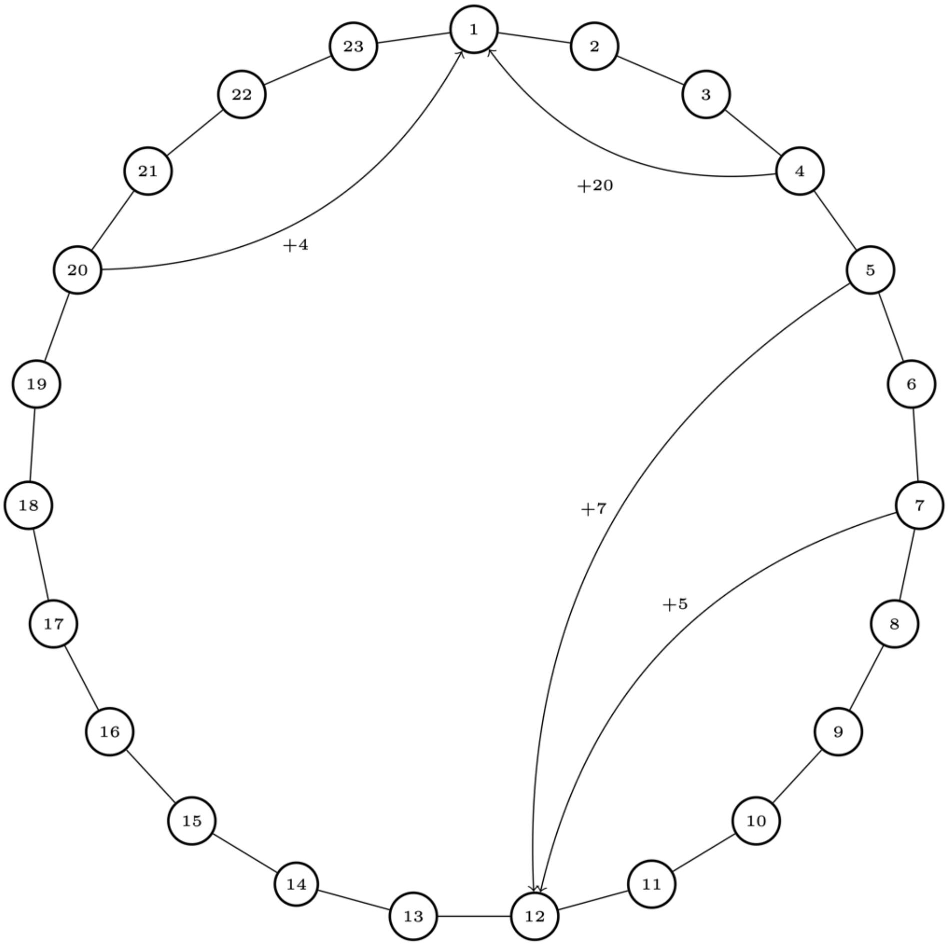

Figure 5. The cyclic group of order 23

In looking at *Figure 5*, for instance, nobody (except a machine maybe) would want to try all possible pairs of additions in order to convince him or herself that the commutative law holds for the cyclic group of order 23 (where indeed $5+7=7+5=12$ and also, say, $20+4=4+20=1$). Introspectively, it appears that we simply see (or perhaps perceive or feel) that the commutative law holds for the entire structure (be it finite or infinite) at first glance, so to speak, or when a couple of instances of the law are shown to us.[15] Especially in the case of universally quantified formulae whose truth has to be checked in infinite structures, a sense of puzzlement thus persists as to how this simple account of the working of intuition could function. On the one hand, it seems that understanding that a formula is true in this framework simply means to check whether the "facts" such formula "speaks about" are true of the mathematical structure in question.[16] In the finite case, intuition would be reducible to a completely spatial and ultimately sensible feature. On the other hand, in the case of complicated or infinite structures, we cannot in general just simply check directly these facts in all possible instances which are captured by the formula. The problem of the sensible nature or origin of intuition thus remains open. Nonetheless, we may delineate a series of (provisional and arguably somewhat tentative) arguments as to the sensible nature or origin of mathematical intuition.

First, since model-theoretic structures are essentially geometric objects, it seems to be possible to conclude that intuition regarding them has also, if not a spatial, at least a geometrical nature (see Gödel's discussion below).[17,18] Second, this conclusion is corroborated by the (perhaps arguable) empirical evidence that people (i.e., students, teachers, and researchers) seem to find it easier to understand, learn, teach, and discover facts about mathematical theories that are directly related to their own sensible experience. Third, while one can certainly do mathematics

---

[15] In other words, following Fischbein (2002), intuition enjoys a feature of "globality" (p. 120) which appears to affect specifically, though not solely, visual representations and which results in what he called a "panoramic account" (*ibid.*) of spatial/visual arrangements of objects and events.

[16] In the Tarskian semantic sense (see Marker, 2002, Chapter 1).

[17] Cf. Chudnoff (2014, p. 183).

[18] Notice however that, generally speaking, intuition does not have to refer to a particular mathematical structure known in advance (though perhaps already existing in advance, a Platonist perhaps would argue), nor it has to be tied to some particular formal language fixed in advance. On the contrary, it is through axioms and reflection upon them that mathematicians form a better and better idea of the structures they are studying, often not having in advance a definition nor a precise notion of them, but only a somewhat blurry idea which they hope to make more precise via formalization. This is also true when one learns mathematics, even within the most axiomatized fields: it is almost always useful to have in advance an intuitive naïve idea of what one is going to study and only then proceed to sharpen this by introducing formalization. Such preliminary naïve idea which precedes formalization is usually referred to in mathematical logic as "the intended model" of the theory, a concept often appearing in the first pages of advanced texts where authors wants to provide their readers with some preliminary intuitive idea of the matter they are going to discuss. It is arguably in this extended sense that Gödel's conception of intuition as directed phenomenological reflection (see the following section) possesses most force. Nonetheless, one could argue that the intended model also always has a somewhat (blurry or fuzzy perhaps) geometrical or spatial nature, and hence the tentative argument above would apply also in this case.

without reference so some intuitively familiar structure (i.e. in some counterintuitive structure, or worse, without reference to some structure, i.e., in a purely syntactical sense), every mathematician would agree that it is hard to fruitfully proceed in a completely formalistic study of mathematics, and that (at least partial) reference to some particular spatial or geometrical structure (in turn partially or locally sensibly apprehensible) helps a great deal in advancing any mathematical endeavour.[19] Fourth, this fact is also reflected in the history of mathematics,[20] which appears to proceed from the description of familiar structures coming from everyday sensory experience up to the abstract and less spatially related theories.[21]

This problem is echoed in the influential treatise *Intuition in science and mathematics* by psychologist Efraim Fischbein. He affirmed that "the basic source of intuitive cognitions is the experience accumulated by a person in relatively constant conditions" (2002, p. 85, emphasis deleted), but he nevertheless conceded that "There is little systematic evidence available supporting" the fundamental experiential nature of intuition (*ibid*.). Furthermore, he claimed that many mathematical concepts (and in particular those pertaining to infinite collections) are beyond what is "visually and behaviorally representable" (p. 96), while nonetheless being "rationally valid" and "meaningful on logical grounds" (*ibid.*). This is because "Their logic is not our logic, which is rooted in practical experiences" (p. 85). Fischbein however suggested that, a grasp of these concepts can be developed by intellectual symbolic manipulation as the (arguably only, he seems to imply) means through which humans can access them.

None of the above arguments is of course conclusive and the problem remains open, as said. Indeed, a sense of mismatch between our characterization of the sensible and the nature of mathematical intuition persists, especially in the infinite case. A discussion of intuition along similar lines is implicitly present in the works of Poincaré who put a strong emphasis on the link between sensible experience and mathematical practice in geometry and analysis.[22]

[19] "The axioms of geometry are "not arbitrary but sensible statements which are, in general, induced by space perception and are determined as to their precise content by expediency" To justify the non-Euclidean axioms Klein pointed out that visualization requires the Euclidean parallel axiom only within certain limits of accuracy" (Felix Klein, 1908 in Morris Kline, 1982, p. 288).

[20] In the words of Michael Atiyah, 1966 Fields Medal, "Broadly speaking we can say that mathematics arose from the practical problems associated with counting, geometry and physics in that order. The development of an adequate concept of number and a proper understanding of its relation to geometry and physics was the first major preoccupation of mathematicians over centuries" (1988, p. 264)

[21] Notice that, contrary to this, there have of course been famous cases in which abstract theories have found a "spatial" application much later. For a counterargument, see however the historical discussion of Kline (1982), which concludes as follows. "What does often happen is that good mathematics, originally motivated by physical problems, finds new application that were not foreseen. Thereby mathematics pays its debt to science. Such uses are to be expected. Are we surprised to find that the hammer which may have been invented originally to crush rocks, may also be used to drive nails into wood? The unexpected scientific uses of mathematical theories arise because the theories are physically grounded to start with and are by no means due to the prophetic insight of all-wise mathematicians who wrestle solely with their souls. The continuing successful use of these creations is by no means fortuitous" (pp. 292–295).

[22] Although even Poincaré had problems with the acceptance of *actual* infinities (see Feferman, 2005, p. 598).

However, Poincaré had doubts on the essentially sensible nature of intuition, specifically with regard to the field of abstract arithmetic. In the end, he left the question of determining the nature of "the intuition of pure number" (1907, pp. 24) and its difference from sensible intuition to the psychologist and the philosopher. Nevertheless, Poincaré seemed keen to affirm this difference on the basis, apparently, of a hard-to-dispel introspective doubt.

> I have said how much the intuition of pure number, whence comes rigorous mathematical induction, differs from sensible intuition […] Is the abyss which separates them less profound than it at first appeared? Could we recognize with a little attention that this pure intuition itself could not do without the aid of the senses? This is the affair of the psychologist and the metaphysician and I shall not discuss the question. But the thing's being doubtful is enough to justify me in recognizing and affirming an essential difference between the two kinds of intuition; they have not the same object and seem to call into play two different faculties of our soul; one would think of two search-lights directed upon two worlds strangers to one another. (*ivi*, pp. 24–25)

5. Gödel and intuition

An extreme position on the problem of the nature and origin of intuition is present in the work of Gödel who, in a sense, radicalized Poincaré's dualistic stance. Gödel, a convinced Platonist, thought of intuition as being very akin to (but different and not strictly related to) sensible intuition, the former having as its objects abstract concepts and the latter having as its object the objects of the physical world. He made his position particularly clear in his 1964 paper *What is Cantor's continuum problem?*, where he analyzed the status of the famous Continuum Hypothesis, a statement which is provably independent from the other axioms of set theory (as Euclid's fifth postulate is independent from the other axioms of geometry).

> In geometry, e.g., the question as to whether Euclid's fifth postulate is true retains its meaning if the primitive terms are taken in a definite sense, i.e., as referring to the behavior of rigid bodies, rays of light, etc. The situation in set theory is similar; the difference is only that, in geometry, the meaning usually adopted today refers to physics rather than to mathematical intuition and that, therefore, a decision falls outside the range of mathematics. On the other hand, the objects of transfinite set theory […] clearly do not belong to the physical world, and even their indirect connection with physical experience is very loose (owing primarily to the fact that set-theoretical concepts play only a minor role in the physical theories of today). (Gödel, 1964, p. 267)

As seen, given the fact that from any mathematical theory "arise" sentences not decidable from the theory itself (by the second incompleteness theorems), the only way to found mathematical knowledge is through human intuition which establishes which of the undecidable sentences (e.g. the continuum hypothesis or its negation) may be considered true with respect to the theory we want to develop. For Gödel, set theory makes the features of intuition particularly evident.

> *But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true.* I don't see any reason why we should have less confidence in this kind of perception, i.e. in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them, and, moreover, to believe that a question not decidable now has meaning and may be decided in the future. The set-theoretical paradoxes are hardly any more troublesome for mathematics than deceptions of the senses are for physics. That new mathematical intuitions leading to a decision of such problems as Cantor's continuum hypothesis are perfectly possible was pointed out earlier […] (Gödel, 1964, p. 268, emphasis added)

Hence, Gödel's view on intuition is somewhat classical and quite simple to state: it seems that we have a sort of "noetic eye" which is able to perceive directly properties of mathematical structures, analogically as to the way the physical eye is able to perceive directly the physical properties of bodies. Furthermore, while the faculty of intuition itself is fallible (for it may lead to paradoxes),[23] Gödel argued that it is always possible to refine it up to the point of eventually deciding the problem of whether the Continuum Hypothesis (or other undecidable statements) is true or false.

> It should be noted that mathematical intuition need not be conceived of as a faculty giving an immediate knowledge of the objects concerned. Rather it seems that, as in the case of physical experience, we form our ideas also of those objects on the basis of something else which is immediately given. Only this something else here is not, or not primarily, the sensations. That something besides the sensations actually is immediately given follows (independently of mathematics) from the fact that even our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations, e.g., the idea of object itself, whereas, on the other hand, by our thinking we cannot create any qualitatively new elements, but only reproduce and combine those that are given. Evidently the "given" underlying mathematics is closely related to the abstract elements contained in our empirical ideas. It by no means follows, however, that the data of this second kind, because they cannot be associated with actions of certain things upon our sense organs, are something purely subjective, as Kant asserted. Rather they, too, may represent an aspect of objective reality, but, as

[23] Elsewhere, Gödel remarked that "our knowledge of the world of concepts may be as limited and incomplete as that of ⟦the⟧ world of things. It is certainly undeniable that this knowledge, in certain cases, not only is incomplete but even indistinct. This occurs in the paradoxes of set theory, which are frequently alleged as a disproof of Platonism, but, I think, quite unjustly. Our visual perceptions sometimes contradict our tactile perceptions, for example, in the case of a rod immersed in water, but nobody in his right mind will conclude from this fact that the outer world does not exist" (Gödel, 1951, p. 321).

> opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality. (*ibid.*)

Thus, Gödel's argument for the existence of abstract object ultimately rests on the internal sensation of the existence of such objects which parallels the internal sensation of the existence of physical objects (see Poincaré's remark quoted at the end of the previous section). Indeed, for Gödel, the existence of the world of mathematical objects (outside or inside us), is equally (un)certain as the existence of the physical world, because "the question of the objective existence of the objects of mathematical intuition […] is an exact replica of the question of the objective existence of the outer world" (*ibid.*). In Kantian terms, Gödel maintained that, as the certainty of the existence of the physical world arises from an internal sensation, a similar internal sensation would indicate that the mathematical world is also objectively existent and distinct from the physical. That the two worlds are different is more evident, he argued, if one considers the objects of set theory rather than the objects of geometry. Indeed, it is perhaps possible to found geometry in a non-syntacticist or conventionalist way, simply by taking its primitive objects (points, lines) as referring to objects of the physical world. On the contrary, when one considers set theory, no obvious connection with the physical world can be inferred. For Gödel, then, the easiest way to explain our intuition of of the abstract objects of set theory is simply to postulate their existence as elements of the pure Platonic part of objective reality (whose other constituent part is of course the sensible physical reality, see Cassou-Noguès, 2015). Granted this, intuition would simply be the faculty that allows us to gaze into this Platonic portion of reality. Thus, Gödel rejects the sensible nature and origin of mathematical intuition and this rejection stems from the dual nature of his overall ontology: the physical and the conceptual are separate and hence sensible and pure intuition are separate too.

Further details on the nature of intuition can be extracted from Gödel's further thoughts on the problem of this faculty's trainability, addressed in the unpublished paper *The modern development of the foundations of mathematics in the light of philosophy* (dating around 1961).

> *In what manner*, however, *is it possible to extend our knowledge of these abstract concepts*, i.e., to make these concepts themselves precise and to gain comprehensive and secure insight into the fundamental relations that subsists among them, i.e., ⟦into⟧ the axioms that hold for them? Obviously not, or in any case not exclusively, by trying to give explicit definition for concepts and proofs for axioms, since for that one obviously needs other undefinable abstract concepts and axioms holding for them. […] *The procedure must* thus *consist*, at least to a large extent, *in a clarification of meaning* that does not consist in giving definitions. Now in fact there exists today the beginning of a science which claims to possess a systematic method for such a clarification of meaning, and that is the phenomenology founded by Husserl. Here clarification of meaning consists *in focusing more sharply on the concepts concerned by directing our attention in a certain way, namely, onto our own acts in the use of these concepts, onto our powers in carrying out our acts, etc.* But one must keep clearly in mind that this phenomenology is not a science in the same sense as the other sciences. Rather it is [or in any case should be] a procedure or technique that should produce in us a

> new state of consciousness in which we describe in detail the basic concepts we use in our thought, or grasp other basic concepts hitherto unknown to us. I believe there is no reason at all to reject such a procedure at the outset as hopeless. (Gödel, 1961, p. 383, emphasis added)

Gödel's tentative hint for sharpening our intuition of true mathematical propositions is thus to resume Husserl's original phenomenological procedure, which may be arguably regarded as a form of introspection or meditation on one's own conceptual practices. "Available evidence indicates that from 1959 on Gödel studied Husserl's work carefully for a number of years" and that he regarded the *Cartesian Meditations* as the closest to real phenomenology" (Wang, 1996, p. 164). Thus, for Gödel, phenomenology is the key to unravelling the mysteries of pure intuition of Platonic objects: it is the true heir to Kant's critical endeavour.[24]

> I would like to point out that this intuitive grasping of ever newer axioms that are logically independent from the earlier ones […] agrees in principle with the Kantian conception of mathematics. […][25] the whole phenomenological method, as I sketched it above, goes back in its ⟦central⟧ idea to Kant, and what Husserl did was merely that he first formulated it more precisely, made it fully conscious and actually carried it out for particular domains (p. 385).

6. Turing's proposal

What was Turing's opinion on the problem of the sensibility of intuition? He never addressed this matter explicitly. Nevertheless, in his most famous 1950 essay *Computing machinery and intelligence* he focused again the question of the mechanizability of human reasoning and pointed to the path that, in his opinion, one would have to follow in order to achieve this objective. The engineer following such path would have to intersect the pedagogue's and the psychologist's path.

> In the process of trying to imitate an adult human mind we are bound to think a good deal about the process which has brought it to the state that it is in. We may notice three components
>
> a) The initial state of the mind, say at birth,
>
> b) The education to which it has been subjected,

[24] For a discussion of Gödel's views on phenomenology see Tieszen (1992).

[25] The omitted passage reads: "Kant asserted that in the derivation of the geometrical theorems we always need new geometrical intuitions, and that therefore a purely logical derivation from a finite number of axioms is impossible. This is demonstrably false [by Tarski's theorem?] However, if in this proposition we replace the term "geometrical" by "mathematical" or "set-theoretical", then it becomes a demonstrably true proposition I believe it to be a general feature of many of Kant's assertions that literally understood they are false but in a broader sense contain deep truths" (*ibid.*).

> c) Other experience, not to be described as education, to which it has been subjected.
>
> In the process of trying to imitate an adult human mind we are bound to think a good deal about the process which has brought it to the state that it is in […] Instead of trying to simulate an adult mind, why not rather try to produce one which simulates the child's? If this were then subjected to an appropriate course of education one would obtain the adult brain. […] We have thus divided our problem into two parts. The child-programme and the education process. (Turing, 1950, pp. 455–456) [26]

These general remarks resemble the later taxonomy of intuitions based on their origin formulated by Fischbein in the aforementioned treatise *Intuition in science and mathematics*, where, building on the work of Jean Piaget, he distinguished between primary and secondary intuition. According to Fischbein, primary intuitions are those which develop independently from systematic instruction on the basis of personal experience. They are produced by the "natural, normal experience of an individual" (Fischbein, 2002, p. 68). These would include component c) of Turing's operational classification above as acting on the given material body substratum which constitutes component a) of Turing's classification. Secondary intuitions, on the other hand, for Fischbein, are learnt by adequate instruction and can contradict primary intuitions. These would include component b) of Turing's classification, again as acting on component a). Moreover, "secondary intuitions may present various degrees of abstraction, sophistication and complexity" (p. 69), also encompassing what traditionally have been called "pure" abstract intuitions of infinite mathematical structures. Nonetheless, Fischbein stressed, neither primary nor secondary intuitions are innate or *a priori* in the traditional sense. Both types of intuition are learnt by the individual as the product of a practice in some activity, be it "natural" or "artificial".

Turing, in his endeavour of mechanizing human intelligence, somewhat anticipating the later psychologists' path, was thus advocating for the careful study of children intelligence and its development through education, as opposed to the development of expert systems (which try

[26] Turing's passage appears to be echoed in the brief sketch of the development of the child's mathematical intuition appearing in Gödel's discussion of the distinction between the sensory and the conceptual (in his 1961 paper mentioned in the previous section). "If one considers the development of a child, one notices that it proceeds in two directions: it consists on the one hand in experimenting with the objects of the external world and with its ⟦own⟧ sensory and motor organs, on the other hand in coming to a better and better understanding of language, and that means – as soon as the child is beyond the most primitive designating ⟦of objects⟧ – of the basic concepts on which it rests. […] Now one may view the whole development of empirical science as a systematic and conscious extension of what the child does when it develops in the first direction. […] That makes it thus seem quite possible that a systematic and conscious advance in the second direction will also far exceed the expectations one may have a priori. In fact, one has examples where, even without the application of a systematic and conscious procedure, but entirely by itself, a considerable further development takes place in the second direction, one that transcends "common sense. Namely it turns out that in the systematic establishment of the axioms of mathematics, new axioms, which do not follow by formal logic from those previously established, again and again become evident" (Gödel, 1961, p. 383–384).

to mimic the adult's mind ability in one specific task, such as playing chess) that was starting to gain traction as the paradigmatic direction in the emerging field of artificial intelligence.

> We may hope that machines will eventually compete with men in all purely intellectual fields. But which are the best ones to start with? Even this is a difficult decision. Many people think that a very abstract activity, like the playing of chess, would be best. It can also be maintained that it is best to provide the machine with the best sense organs that money can buy, and then teach it to understand and speak English. This process could follow the normal teaching of a child. Things would be pointed out and named, etc. Again I do not know what the right answer is, but I think both approaches should be tried. (Turing, 1950, p. 460)

Indeed, in 1948 Turing submitted a report to the National Physical Laboratory in London where he advanced the proposal of constructing what he there called "unorganized machines" (Turing, 1948, p. 421). This class of machines are perhaps the first theoretical example of what today we call "artificial neural networks"[27] which can be "educated", so to speak, to perform various human tasks by means of an appropriate training.[28]

> Turing even went on to propose the idea of giving such a learning machine wheels, arms and camera eyes so that it was mobile enough to find out things for itself in the world. At the time, such prognostications caused outcry at NPL, some claiming that "Turing is going to infest the countryside with a robot which will live on twigs and scrap iron!". (Webster, 2012, p. 36)

However, already in 1948, Turing apparently was frustrated by the absence of approval from his co-workers and the lack of available computing power which determined his inability to further explore the potential of his unorganized machines (this latter apparently was a fundamental reason for the general abandonment of the neural nets program in the 1960s, see Dreyfus & Dreyfus, 1988, p. 24). Nonetheless, the examination of Turing's proposals of endowing machines with body organs leads us towards the direction of concluding that Turing's earlier articulated notion of intuition emerges as being of an essentially spatial or sensible character, tied to the representation of one's own body into space.

Moreover, Turing appears to be saying more than this: very ahead of his time, he was seemingly here promoting the development of (embodied) cognitive models of the human mind. Indeed, for Turing, since there is no way to circumvent the need for intuition by other means (he discusses the issue at length in the previous pages), the obvious step for imitating the human mind, is trying to give a machine "the best sense organs that money can buy" as the only way for hard-wiring primary intuitions (in Fischbein's classification) into machines and letting them experience the spatial sensible world. Secondary intuition (again in Fischbein's classification)

[27] See Dreyfus & Dreyfus (1988).

[28] "Turing may have been the first to suggest using randomly connected networks of neuron-like nodes to perform computation, and proposed the construction of large, brain-like networks of such neurons capable of being trained as one would teach a child" (Webster, 2012, p. 35).

would then be developed in the same way they are developed by humans: through an ordinary process of education.

A surprising analogy thus appears to connect Turing's insight with the more recent theories of embodied cognition, which have stressed the essential significance of the human body to thought in general and to mathematics in particular. For instance, in the book *Where mathematics comes from* by linguist George Lakoff and psychologist Rafael Núñez, a strong relation between human experience and mathematics is argued as a result of various empirical findings and theoretical considerations (see Edwards & Robutti, 2014).[29] The leading idea in this theory (sometimes called the theory of embodied mathematics) is the crucial role of the mediation of embodied spatiality in the development of logical and mathematical conceptualizations, with respect to both those pertaining to finite as well as those pertaining to infinite objects.

> Human mathematics is embodied, it is grounded in bodily experience in the world [...] not purely subjective […] not a matter of mere social agreement […] It uses the very limited and constrained resources of human biology and is shaped by the nature of our brains, our bodies, our conceptual systems, and the concerns of human societies and cultures (Lakoff & Núñez, 2000, pp. 348 –365).

Notice that "Turing did not consistently defend the centrality of embodiment" (Cappuccio, 2016, p. 485), often reverting to (what would now be considered) a more traditional cognitivist understanding of thought focused on symbolic manipulation and formal rules in order to explain or mimic human cognitive functions. Nonetheless, he highlighted how

> the emergence of these functions (i.e., the developmentally necessary preconditions of their genesis and actual implementability) must be rooted into an embodied process of learning based on situated interactions and unprincipled exploration. (Cappuccio, 2016, p. 485)

Moreover, Turing acknowledged that the body is not merely an extension of a mind operating in an abstract void but a necessary component of the mind itself. Furthermore, Turing emphasized the fact that dynamic interaction with the environment is essential for developing adaptive and intelligent behaviors. Indeed, he recognized that "an intelligent system requires a fine-grained calibration to contextual circumstances, and that only a living organic body can achieve it through a history of complex adaptive interactions with the world and with other organs" (*ivi*, p. 486), thus anticipating "that the body is the fundamental constraint/organizer/regulator in all the evolutionary and developmental processes of adaptation" (*ivi*, p. 487).

[29] See Lolli (2000) for a critique of the general philosophical claims of Lakoff and Núñez. See also Schiralli & Sinclair (2003) for a critique from an embodied perspective.

7. Conclusion

While Turing's approach to the problem of mathematical cognition does not in itself counter philosophically the traditional Platonist view of mathematics, his overall work since the beginning, appears to mark

> the epistemological break between idealism and a materialism in mathematics. Prior to Turing it was hard to get away from the idea that through mathematical reason, the human mind gained access to a higher domain of Platonic truths. (Cockshott, 2012, p. 1; see also Hodges, 2012)

Furthermore, Turing's materialist stance with regard to mathematical intuition and to cognition in general is not the simplistic synctacticist stance criticized by Gödel nor a naïve form of conventionalism (see Ernest, 2023 quoted in Note 6). Thus, Turing's analysis of the notion of intuition as the seemingly irreducible faculty stemming from the impossibility of mechanization of mathematical reasoning helps us in situating precisely where metamathematically-informed materialist evaluations and criticism have to be carried out and directed both on philosophical and educational ground.[30,31]

Of course, I stress it again, neither Turing's later tentative argument nor more recent and more detailed embodied accounts of human mathematical reasoning are conclusive in refuting dualist accounts of intuition, nor in grounding its origin and development on firm materialist foundations.[32] Nevertheless, Turing's later views on the nature of intelligence align with the idea that mathematical intuition emerges from dynamic embodied interactions with the physical environment, a view which appear to starkly contrast with Gödel's belief in the independent abstract existence of mathematical objects passible to be apprehended by a noetic suprasensible mental faculty. Further elaborations on Turing's ideas could then serve to ground an understanding of mathematics which could possibly put into question the ideology of the purity stemming from its traditional perceived detachment from the physical, together with the enormous symbolic power derived from this ideology (see Dowling, 2002; Ernest, 2020, 2021;

---

[30] Or whether perhaps even extreme Platonist views on mathematics such as Gödel's (and his tentatively Kantian phenomenological remarks) can ultimately be dialectically accommodated in part with materialist perspectives. Cf., e.g., the phenomenological (Hegelian) embodied perspective of Radford (2014). Cf. also incidentally the conversation between Anna Sfard and Paul Ernest (2017) on the possibility of accounting monistically for the seemingly dualistic nature of the human (mathematical) experience, partly related to the difficulty of wholly encompassing the latter (in particular the introspective, non-verbal, and seemingly incommunicable first-person experiences) under theoretical conceptual constructs such as "dialogue", "conversation", "speech", or "discourse" (unless they are understood in an all-encompassing, and hence perhaps theoretically void, sense). The difficulty of reducing intuition to formalization as connected to the apparently unavoidable gap between the two discussed by Turing and Gödel could be perhaps read as a particular instance of this more general phenomenon.

[31] For instance, I would maintain that the Lakatosian-inspired tradition of stressing on the fallibility of mathematical intuition bears little force on countering (the most sophisticated forms of) Platonism for good (see Note 23; however cf. Ernest, 2014 for a discussion; cf. also Ernest, 2016).

[32] Similarly, the exploratory arguments of Section 4 on the sensible nature of mathematical intuition appear to not be decisive in this respect, as said.

Beccuti, 2025), as well as, more ambitiously, "to explain how abstract, formal mathematical ideas can emerge from concrete sensory-motor experience" (Schiralli & Sinclair, 2003, p. 79; Presmeg, 2002; de Freitas & Sinclair, 2013, Edwards & Robutti, 2014; Radford et al., 2017; Sriraman & Wu, 2020).

As we have seen, Turing's perspective can be put into fruitful dialogue with major recent results of psychology and cognitive science which have proved to be vastly influential, among other fields, in mathematics education research (i.e., Kahneman, Dehaene, Fischbein, Lakoff and Núñez). Retrieving and expanding Turing's reflections on mathematical cognition could thus serve to fruitfully complement the perspectives of those scholars who do not adhere a radical formalist or naïve conventionalist account of mathematical ontology, but are nonetheless prone to reject a dualist general epistemology involving noetic access to a Platonic world of ideas. To these thinkers Turing's standpoint will appear, I believe, as a most promising perspective to be put into dialogue with materialist, human-centered, and embodied philosophies of mathematics and mathematics education.

References

Atiyah, M. (1988). *Collected works*: *Vol. 1*. Clarendon Press.

Beccuti, F. (2018). La disgiunzione di Gödel. *APHEX Rivista Italiana di Filosofia Analitica, 18*. pp. 1–22. Accessed on the 14th of August 2024 at https://www.aphex.it/la-disgiunzione-di-godel/.

Beccuti, F. (2025). Ideology and mathematics education: A rebound. *For the Learning of Mathematics* (in press).

Cappuccio, M. L. (2016). The seminal speculation of a precursor: Elements of embodied cognition and situated AI in Alan Turing. In V. C. Müller (Ed.) *Fundamental issues of artificial intelligence* (pp. 479–496). Springer.

Cassou-Noguès, P. (2015). *Les démons de Gödel. Logique et folie: Logique et folie*. Média Diffusion.

Chudnoff, E. (2014). Intuition in mathematics. In B. Held and L. Osbeck (Eds.) *Rational intuition* (pp. 174–191). Cambridge University Press.

Cockshott, P. (2012). Turing: the irruption of materialism into thought. Oxford University Press blog. Available at https://blog.oup.com/2012/06/turing-the-irruption-of-materialism-into-thought/.

Cockshott, P., Mackenzie, L. M., & Michaelson, G. (2012). *Computation and its limits*. Oxford University Press.

de Freitas, E. & Sinclair, N. (2013). New materialist ontologies in mathematics education: the body in/of mathematics. *Educational Studies in Mathematics, 83,* 453–470.

Dehaene, S. (1997). *The number sense*. Oxford University Press.

Dehaene, S., Bossini, S., & Giraux, P. (1993). The mental representation of parity and number magnitude. *Journal of Experimental Psychology, 122*, 371–396.

Detlefsen, M. (2005). Formalism. In S. Shapiro (2005, pp. 236–317).

Dowling, P. (2002). *The sociology of mathematics education: Mathematical myths/pedagogic texts*. Routledge.

Dreyfus, H. & Dreyfus, S. E. (1988). Making a mind versus modeling the brain: Artificial intelligence back at a branchpoint. *Daedalus, 117*(1), 15–43.

Edwards, L. D. & Robutti, O. (2014). Embodiment, modalities, and mathematical affordances. In L. D. Edwards et al. (Eds.), *Emerging perspectives on gesture and embodiment in mathematics* (pp. 1–23). Information Age Publishing.

Evans, J. (2012). Dual-process theories of deductive reasoning: Facts and Fallacies. In K. Holyoak and R. Morrison (Eds.) *The Oxford Handbook of Thinking and Reasoning* (pp. 115–133). Oxford University Press.

Evans, J. & Stanovich, K. (2013). Dual-process theories of higher cognition: Advancing the debate. *Perspectives on Psychological Science, 8,* 223–241.

Ernest, P. (2014). Certainty In Mathematics: Is There A Problem?. *Philosophy of Mathematics Education Journal*, *28*, Article 9. Accessed on the 14th of August 2024 at https://www.exeter.ac.uk/research/groups/education/pmej/pome28/index.html.

Ernest, P. (2016). The problem of certainty in mathematics. *Educational Studies in Mathematics, 92*, 379–393.

Ernest, P. (2020). The ideologies of purity and neutrality and the ethics of mathematics. *Philosophy of Mathematics Education Journal, 36*, Article 2. Accessed on the 14th of

August 2024 at https://www.exeter.ac.uk/research/groups/education/pmej/pome36/index.html.

Ernest, P. (2021). Mathematics, ethics and purism: an application of MacIntyre's virtue theory. *Synthese, 199*, 3137–3167.

Ernest, P. (2023). The problems of necessity and authority in mathematics and education. *Journal for Theoretical & Marginal Mathematics Education, 2*(1), Article 0205. Accessed on the 14th of August 2024 at https://www.cppi.me/journal/vol2.

Ernest, P. & Sfard, A. (2017). A dialogue on dialogue. *Philosophy of Mathematics Education Journal, 33*, Article 1. Accessed on the 14th of August 2024 at https://www.exeter.ac.uk/research/groups/education/pmej/pome33/index.html.

Feferman, S., (2005). Predicativity. In S. Shapiro (2005, pp. 590–624).

Fischbein, E. (2002). *Intuition in science and mathematics: An educational approach*. Kluwer Academic Publishers.

Gigerenzer, G. (2011). Personal reflections on theory and psychology. *Theory & Psychology, 20*, 733–743.

Gödel, K. (1951). Some basic theorems on the foundations of mathematics and their implications. In K. Gödel (1995, pp. 304–323).

Gödel, K. (1953). Is mathematics syntax of language?. In K. Gödel (1995, pp. 334–362).

Gödel, K. (1961). The modern development of the foundations of mathematics in the light of philosophy. In K. Gödel (1995, pp. 374–387).

Gödel, K. (1964). What is Cantor's continuum problem?. In K. Gödel (1990, pp. 252–270).

Gödel, K. (1972). Some remarks on the undecidability results. In K. Gödel (1990, pp. 305–306).

Gödel, K. (1990). *Collected works. Volume II* (S. Feferman, et al. Eds.). Oxford University Press.

Gödel, K. (1995). *Collected works. Volume III* (S. Feferman, et al. Eds.). Oxford University Press.

Goldfarb, W. (1995). Introductory note to *1953/9. In K. Gödel (1995, pp. 324–334).

Hahn, H. (1933/1980). The crisis in intuition. In B. McGuinness (Ed.) *Empiricism, Logic and Mathematics* (pp. 73–103). Springer.

Hilbert D., (1902). Mathematical problems: Lecture delivered before the International Congress of Mathematicians at Paris in 1900. *Bullettin of the American Mathematical Society, 8,* 437–479.

Hodges, W. (1998). Turing's philosophical error?. In L. J. Landau & J. G. Taylor (Eds.) *Concepts for Neural Networks. Perspectives in Neural Computing* (pp. 147–169). Springer.

Hodges, W. (2012). *Alan Turing: The Enigma*. Princeton University Press.

Kahneman, D. (2011). *Thinking fast and slow*. Farrar Straus & Giroux.

Kant, I. (1781/1998). *Critique of pure reason*. Cambridge University Press.

Kennedy, J. (2020a). A philosophical argument about the content of mathematics. Supplement to Kennedy, J. (2020). Kurt Gödel. In E. N. Zalta (Ed.) *The Stanford Encyclopedia of Philosophy* (Winter 2020 Edition). Accessed on the 14th of August 2024 at https://plato.stanford.edu/archives/win2020/entries/goedel/content-mathematics.html.

Kennedy, J. (2020b). Did the Incompleteness Theorems Refute Hilbert's Program?. Supplement to Kennedy, J. (2020). Kurt Gödel. In E. N. Zalta (Ed.) *The Stanford Encyclopedia of Philosophy* (Winter 2020 Edition). Accessed on the 14th of May 2024 at https://plato.stanford.edu/archives/win2020/entries/goedel/incompleteness-hilbert.html.

Kline, M. (1982). *Mathematics: The loss of certainty*. Oxford University Press.

Lakoff, G. & Núñez, E. R. (2000). *Where mathematics comes from*. Basic Books.

Lolli, G. (2000). *Da dove viene la matematica: Recensione di G. Lakoff, R. E. Nuñez: Where mathematics comes from*. Accessed on the 15th of August 2024 at https://web.archive.org/web/20120804004046/https://homepage.sns.it/lolli/articoli/LakNun.pdf.

Marker, D. (2002). *Model theory: An introduction*. Springer.

Osman, M. (2013). A Case Study: Dual-Process Theories of Higher Cognition–Commentary on Evans & Stanovich (2013). *Perspectives on Psychological Science, 8*, 248–252.

Piccinini, G. (2003). Alan Turing and the mathematical objection. *Minds and Machines*, *13*, 23–48.

Poincaré, H. (1907). *The value of science.* Dover Publications.

Presmeg, G. N. (2002). Mathematical idea analysis: a science of embodied mathematics. *Journal for Research in Mathematics education, 33(1)*, 59–63.

Radford, L. (2014). Towards an embodied, cultural, and material conception of mathematics cognition. *ZDM Mathematics Education, 46*, 349–361.

Radford, L, Arzarello, F., Edwards, L., & Sabena, C. (2017). The Multimodal Material Mind: Embodiment in mathematics education. In J. Cai (Ed.) *Compendium for research in mathematics education* (pp. 700-721). National Council of Teachers of Mathematics.

Schiralli, M. & Sinclair, N. (2003). A constructive response to 'Where mathematics comes from'. *Educational Studies in Mathematics, 52*(1), 59–91.

Schopenhauer, A. (1813/1974). *On the fourfold root of the principle of sufficient reason*. Open Court Press.

Schopenhauer, A. (1819/1969). *The world as will and representation*. Dover Publications.

Shapiro, S. (Ed.) (2005). *The Oxford Handbook of Philosophy of Mathematics and Logic*. Oxford University Press.

Sriraman, B. & Wu, K. (2020). Embodied cognition. In S. Lerman (Ed.) *Encyclopedia of mathematics education* (pp. 266–268). Springer.

Tieszen, R. (1992). Kurt Gödel and phenomenology. *Philosophy of Science, 49*(2), 176–194.

Turing, A. M. (1937). On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society, s2-42*(1), 230–265.

Turing, A. M. (1939). Systems of logic based on ordinals. *Proceedings of the London Mathematical Society, 45*, 161–228.

Turing, A. M. (1948). Intelligent machinery. In J. Copeland (Ed.) (2004). *The essential Turing* (pp. 410–432). Clarendon Press.

Turing, A. M. (1950). Computing machinery and intelligence. *Mind, 49*, 433–460.

Wang, H. (1996). *A logical journey.* The MIT Press.

Webster, C. S. (2012). Alan Turing's unorganized machines and artificial neural networks. *Evolutionary Intelligence, 5*, 35–43.

Zach, R. (2023). Hilbert's Program. E. N. Zalta & U. Nodelman (Eds.). *The Stanford Encyclopedia of Philosophy* (Winter 2023 Edition). Accessed on the 14th of May 2024 at https://plato.stanford.edu/archives/win2023/entries/hilbert-program/.