MATHEMATICS EDUCATION FOR CITIZENSHIP: POTENTIALS AND REALITIES[1]

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Preliminary abstract

I will depart from a somewhat elaborated understanding of citizenship to discuss which education is needed for citizenship. Based on this framework, I will ask which roles education in mathematics can play for citizenship, but also which special challenges the use of mathematics in our societies poses. Arriving at an idea of what mathematics education for citizenship might entail, I will ask in how far mathematics education does or could fulfil theses expectations.

Introduction

Is mathematics education helpful, maybe even necessary, but perhaps also an obstacle to citizenship? While we may all have an experience and a research study on this topic to share, this article will attempt to provide a more structured answer. We will first develop an understanding of citizenship, mathematics, and mathematics education, before we look for possible, maybe even necessary, connections and frictions. We will attempt to do that from a theoretical discussion of what could be, but we will end with some reflections grounded in empiricism to challenge our findings.

Citizenship

Leydet's (2023) entry on "Citizenship" in the *Standford Encyclopedia of Philosophy* defines the citizen as "a member of a political community who enjoys the rights and assumes the duties of membership" (p. 1). *Citizenship* then is the status of an individual that arises from being a member of such a community. Such a definition gives rise to several questions: What might such a "political community" be? Who can be a citizen there? What rights does the citizen enjoy? What are the duties of a citizen, and how is the fulfilment of those duties ensured?

Already the first two questions are not easy to answer. Citizenship is sometimes confused with nationality although there is a legal difference. States can define who has the rights and duties of a citizen, irrespective of nationality. Until only

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half a century ago, there were European countries that did not accept their female nationals as full citizens. Usually, minors are not considered full citizens. Immigrants are usually not awarded full citizenship, at least not before they acquire the nationality of the country they live in. For our purposes in education, we can simplify this discussion by assuming that every student will eventually become a citizen, either in the country of the school or elsewhere.

When it comes to the rights and duties of citizens, it will be helpful to distinguish between three dimensions of citizenship: legal status, political agency, and identity (Carens, 2000; Kymlicka & Norman, 2000): Each citizen has rights that are protected by law. The right to elect the legislative institution of a state is shared by all liberal democracies, but there may be more rights, for example the right to organise petitions or the right to demand certain information from state institutions. Laws can also define duties of citizenship, such as an obligation to vote, jury duty, and military service.

Political agency arises from the rights of citizens to participate in political decision making. However, most, if not all, of the duties of a citizen in the political realm are not legal but idealistic in nature. If there is no legal obligation to vote, then a citizen can choose not to vote. Even if there is a legal obligation to vote, this does not mean that citizens base their votes on well-informed decisions. The expectation that citizens educate themselves about political questions in order to come to well-informed decisions at the poll cannot be realised legally but stays an ideal in liberal democracies.

Identity is a third dimension because membership in a political community can be a source for defining who one is. In fact, shared aspects of such identities (such as a shared language, a shared space on the map, a shared history) can be regarded as a usual (although not the only possible) psychological foundation of an understanding of citizenship as togetherness (Carens, 2000).

Civic education can then be understood as the aspect of education that affects people’s abilities to perform their rights and duties as citizens or prospective citizens. Such education often takes place in schools but is also facilitated elsewhere, for example, in families, religions, and mass media. Different to the context in schools, where civic education might be a social obligation in a limited period of one’s life, such education can also be understood as an enduring individual project to educate oneself to become more valuable for one’s community, as expressed in the Ancient Greek term *paideia*.

Civic education can be an explicit goal or an implicit effect of social intercourse. Crittenden and Levine (2024) stress that “civic education need not be intentional or deliberate; institutions and communities transmit values and norms without meaning to” (p. 1). Also, civic education does not necessarily facilitate the achievement of the ideals of citizenship; just as well, it can hinder them. Crittenden and Levine (2024) underline that “sometimes people are civically educated in ways that disempower them or impart harmful values and goals" (p. 1).

It is important to realise that the concept of citizenship is inseparably connected to political ideologies. Early ideas of citizenship developed as scholars contemplated on new relationships between the state and its subjects at the transition from monarchies to democracies. Today, scholarship distinguishes competing conceptualisations of citizenship, some more conservative and others more progressive (Leydet, 2023). These conceptualisations provide different answers to basic questions in the field, such as (Crittenden & Levine, 2024): "What responsibilities does a citizen […] have?", "What is the relationship between a good regime and good citizenship?", "Who may decide what constitutes good citizenship?", and "What means of civic education are ethically appropriate?" (pp. 3–4). However, Crittenden and Levine conclude that "these questions are rarely treated together as part of comprehensive theories of civic education" and that "some of these questions have never been much explored by professional philosophers" (p. 4).

As a consequence, we will not be able to base our following discussions on a comprehensive theory of civic education. However, we can take up the thoughts developed above in order to pose questions on connections between mathematics education and citizenship:

1. In how far does a citizen need education in mathematics to pursue the legal rights and duties of citizenship?
2. In how far does a citizen need education in mathematics to live to up the ideals of political agency?
3. In how far does education in mathematics contribute to the identity dimension of citizenship?
4. In how far do the goals of mathematics education align to or contradict the goals of civic education?
5. In how far does the practice of mathematics education align to or stand in conflict with the goals of civic education?

I will not say much about the first and the third question, as I consider the other questions to have more impact. We will pursue the remaining questions after some elaborations on mathematics and its education.

Mathematics and its education

There is no consensus on what mathematics is (Hacking, 2014), but we may depart from an agreement on vital features of what we call "mathematics". Krishnan (2009) lists the following characteristics of academic disciplines:

1) disciplines have a particular object of research […];
2) disciplines have a body of accumulated specialist knowledge referring to their object of research, which is specific to them and not generally shared with another discipline;

3) disciplines have theories and concepts that can organise the accumulated specialist knowledge effectively;
4) disciplines use specific terminologies or a specific technical language adjusted to their research object;
5) disciplines have developed specific research methods according to their specific research requirements; and maybe most crucially
6) disciplines must have some institutional manifestation in the form of subjects taught at universities or colleges, respective academic departments and professional associations connected to it. (p. 9)

While some of these characteristics are obviously fulfilled, for example, the fourth and the last one, already the first characteristic poses problems for mathematics. It does not seem helpful to characterise mathematics by saying that it is used in economy, engineering, and in medicine. It is, however, helpful to conclude that similar mathematics is used in many domains. But what, then, is its “particular object of research”? Would it be fair to say that mathematics deals with abstract concepts? I think not, for every discipline has its own concepts and abstraction is relative: From a specific perspective, even “table” could an abstract concept, but “proton” and “metaphor” clearly are abstract concepts.

A solution could be to look at the third and fifth point first: How is mathematical knowledge organised and how do mathematicians legitimise this knowledge? An interesting observation is that mathematical theories usually do not refer to domains where they may be applied, even though they may arise from such domains. Methodically, such theories have to follow a strict deductive architecture, even though there may be a more or less implicit possibility to validate the usefulness of such a theory in a domain of its applications. This characterisation does not propose that there are any limitations concerning the objects of mathematics other than their suitability to become objects in a deductive theory.

If this is so, then the interplay between mathematical knowledge and its domains of application becomes all the more difficult to explain. On the one hand, it becomes a question why mathematics can be successfully applied at all. Wigner’s (1960) article on “The Unreasonable Effectiveness of Mathematics in the Natural Sciences”, although somewhat naïve from a philosophical point of view, has expressed the amazement of mathematicians and natural scientists alike. A part of this effectiveness can be explained by the fact that mathematics is not as effective as we think, because many applications, however prominent, turn out not to work well (see, e.g., Cartwright, 1983). We may add that it is not possible to apply any mathematics to any domain; only very specific applications turn out to be useful. In the end, their usefulness is an empirical criterion based in the domain of application, so, in a sense, mathematics turns out to be empirical after all.

If we look more closely on applications of mathematics, we can observe that the question is not if mathematics fits or does not fit to the domain. Instead, the question is how the perception of the domain had to be altered in order to allow for an application of mathematics. In order to express it with quadratic equations, physicists began to conceive of free fall in a vacuum long before anyone could actually produce such a vacuum. Consequently, mathematics is not only used to describe objects which exist independently from mathematics. Rather, the application of mathematics always reformats our understanding of such objects. Skovsmose (1994) has called this the *formatting power* of mathematics in the sense that mathematics formats our understanding of the real. This has become all the more clear through a new research focus in mathematics education on *normative modelling* (e.g., Pohlkamp & Heitzer, 2021).

The main goal of *mathematics education* is often conceived as getting students to learn mathematical concepts and procedures. Biesta (2009) speaks of a "learnification" of our conception of education (p. 38). If this were the main function of mathematics education, it would be an extremely ineffective enterprise. Studies from different countries show that people can hardly use mathematical concepts and apply mathematical procedures beyond basic arithmetic operations, once they have left school (Department for Business, Innovation and Skills, 2012; Maaß & Schlöglmann, 2000). Increasingly, mathematical qualification is defined in connection to the ability to apply mathematics to reality (e.g., National Council of Teachers of Mathematics, 2000). Elsewhere, I discuss other functions of mathematics education that have been addressed in the research literature (Kollosche, 2018). Some of them are frequently addressed as explicit goals of mathematics education in policy documents, others are not. Among them are the function to *integrate* students into the social relations and practices in society (such as being a problem solver) and the function to *legitimise* social institutions and power relations (such as the persuasive power of mathematics).

Qualification in mathematics education for citizenship

It is uncontested that some knowledge of and skills in mathematics are necessary to perform the political agency of citizenship. This becomes most obvious when information is required to take part in direct democracy, in elections of representatives, or, more generally, in political discussions. While the concept of *mathematical literacy* might well describe the ability to found judgement in real-world contexts on mathematical knowledge and skills (Niss & Jablonka, 2020), scholarship on mathematical literacy has not always explicated its importance for citizenship. An exception would be Skovsmose's (1994) interpretation of "mathemacy", about which he writes:

Could mathemacy also be used for the purpose of empowerment? Could mathemacy help people to reorganise their views about social institutions, traditions and possibilities in political actions? […] Mathemacy, as a radical construct, has to be rooted in the spirit of critique and the project of possibility that enables people to participate in the understanding and transformation of their society and, therefore, mathemacy becomes a precondition for social and cultural emancipation. (pp. 26–27)

Here, Skovsmose talks about the help of mathematics in envisioning and realising political changes. A different perspective is to acknowledge that mathematics is required to understand the discourses on which political decisions are based. Skovsmose (1994) addresses this question when he states:

The ground for decisions taken by the authorities may be inaccessible to people other than the technicians and the people in charge. Technological development may erode part of the non-formal conditions for democracy, leaving behind only an algorithm for election. That erosion is a real threat to democracy in a highly technological society. But is it possible to secure a critical citizenship in a highly technological society? To find a positive answer to this question is equivalent to conceiving democratic life as possible, in the future as well as the present. (pp. 39–40)

Skovsmose's (1994) solution is to demand reflective knowing in mathematics (pp. 97–114), which I described elsewhere to comprise "recovering the modelled situation in its complexity, addressing problems and uncertainties in the transitions from situations to real models and mathematics models and back, and identifying in which way the mathematical model is formatting reality" (Kollosche & Meyerhöfer, 2021, p. 403). I am afraid that there is hardly any research on such processes of reflection in mathematics education (an exception being Plunger, 2021), although research on normative modelling is certainly informative in this respect (e.g., Pohlkamp & Heitzer, 2021).

When Meyerhöfer and I analysed the mathematics used in the political discourses on the COVID-19 pandemic, we had to conclude that such reflexion for the purpose of allowing political agency is not always possible (Kollosche & Meyerhöfer, 2021). We analysed mathematics that was used in the political discourses in Germany, including the quantification of lethality, the counting of casualties, the reproduction number R, and the model of an exponential growth of infection numbers. In most cases, "we faced ideal concepts, which are easy to define and possible to understand as mathematical laypersons but utterly impractical or even impossible to actually determine" (p. 412). In the end, such concepts had to be redefined mathematically in ways that "cannot be understood by mathematical laypersons to an extent that would allow for the evaluation of such expert knowledge" (p. 414). We concluded "that critical reflection of mathematics cannot be the sole solution to the problem it stood up against", that is agency in political discourses (p. 415).

However, the idea that qualification in mathematics should allow for political agency is not uncontested. In Austria, for example, the focus of secondary level education in mathematics is slowly moving from reflecting on the use of mathematics towards preparing learners for university studies in mathematics (which is pursued by a very few school graduates only). This aligns well with a general consent that mathematics should be unpolitical, which impedes any attempts of teaching and learning to apply mathematics to applications of socio-political relevance.

Eventually, while it was clear that students cannot master all the mathematical knowledge and skills needed to understand the complete variety of political discourses, even the hope that mathematics education could lay a foundation upon which further self-study might allow to understand the mathematics behind selected political discourses, became compromised. Still, what one might hope for is that mathematics education equips learners with a basic understanding of mathematics together with a mathematical habitus of thought, skills of self-learning and critical reflection, and confidence in their own mathematical abilities. This insight does not only put the importance of qualification into perspective, it also shifts our focus on aspects of mathematics education that may eventually turn out to be problematic in our contemporary practice.

Integration through mathematics education for citizenship

The integration function of education refers to the preparation of "adolescents to successfully integrate into the social relations and practices in society" (Kollosche, 2018, p. 293). In his history of the political economy of mathematics education, Neander (1974) points out how the political goals of mathematics education matched the desired habitus of workers at a given time in Modern Germany. For example, problem solving in mathematics education can be understood as an educational answer to new demands on workers in the knowledge society (Dahl, 2014). Skovsmose (2005) suggests that such practices are not isolated from the rest of society but in fact functional:

Could it be that 'normal' students in fact learn 'something', although not strictly speaking mathematics (and certainly not mathematical creativity), and that this 'something' serves an important social function? If we look back again at the 10,000 commandments [that is exercises that students have to solve during their school careers], what do they look like? Certainly, not like any of those tasks with which applied mathematics occupies itself, tasks in which creativity is needed to construct a model of a selected piece of reality. Nor do they look like anything a working mathematician is doing. However, they might have some similarities with those routine tasks, which are found everywhere in production and administration. An accountant has to do sums day after day. A laboratory assistant has to do a series of routine tasks in a careful way. [...] All such jobs do not invite creative ways of using numbers and figures. Instead things have to be handled carefully and correctly in a pre-described way. Could it be that the school mathematics tradition is a well functioning preparation for a majority of students who come to serve in such job-functions? (pp. 11–12)

If we accept that mathematics education serves such an integration function of schooling, then what would citizenship require from it? Connecting to the last paragraph of our thoughts on qualification, we can say that citizenship requires students to learn to look at socio-politically relevant issues from a mathematical perspective, to acquire needed mathematical knowledge and skills themselves, and to reflect on the usage of mathematics in applications. We may first discuss how these aspects are represented in mathematics education research before we turn our focus on school practice.

What does it mean to look at issues from a mathematical perspective? We may say that this involves, among other things, the ability to think in deductively organised structures and to apply such structures to situations in reality.

Learning to think in deductively organised knowledge structures is often addressed under the terms of mathematical reasoning and proof, which constitutes an extensive field of research in mathematics education (see, e.g., Harel & Weber, 2020). Historically, education in mathematical reasoning had long been reserved for elitist circles, and only set out to become a universal standard for mathematics education with the new math movement in the 1960s and 1970s (Kilpatrick, 2012). However, despite numerous proofs that developing skills in mathematical reasoning is possible already in primary school (e.g., Smit et al., 2023), many countries have developed classroom cultures with very limited activities in mathematical reasoning. To give an example, there are schoolbooks sanctioned by the ministry of education for use in grammar schools in Austria which introduce nearly all of the new mathematical knowledge without any validation of why it holds true. Apart from this lack if implementation, it is unclear in how far mathematical reasoning can help to perform citizenship. For example, the naïve view that a politician's argument can be judged on the basis of its deductive logic, is often misled, as discourse, also in academia, uses different styles of reasoning (Crombie, 1994). Also, the desired education in mathematical reasoning usually does not include reflections upon the epistemic limits and drawbacks of this form of reasoning, thus not allowing for the development of what Skovsmose called "reflective knowing". As I have argued elsewhere, it might be worthwhile to position mathematical reasoning within a framework of styles of reasoning in order to locate its place in our attempts to reach objectivity, be it in academia or in socio-political discourses (Kollosche, 2021).

As with mathematical reasoning and proof, a lot of research has been done in the last three decades on mathematical modelling for the classroom (for an overview, see, e.g., Kaiser, 2020). However, it is curious that the modelling processes pursued in such research strongly focus on the production of mathematical models rather than on their evaluation (K. Lengnink, personal communication, Oct. 2022). Thus, this research focusses on preparing students to become appliers of mathematics in complex situations rather than on preparing them to question applications of mathematics in the sense of the reflexion proposed by Skovsmose (1994). It fits to that diagnosis that there is hardly any interest in the structural processes of mathematising real-world problems or interpreting and validating mathematical solutions in this research field, as I have noted elsewhere (Kollosche, 2021). Put into practise, the effect of such an education in mathematical modelling might be that some students become useful appliers of mathematics, which might qualify a few for specific professional tasks, whereas a majority would still not be able to reflect on socio-politically relevant applications of mathematics.

The remaining question would be in how far students learn to acquire needed mathematical knowledge and skills by themselves. Individuals need to know where they can get information, how to distinguish good sources from bad ones, how to deal with problems during their learning processes, where to get support if they need it. All this differs from classroom situations as the learning situation is not pre-designed by the teacher or other educators and as there are no teachers or other students for interaction. Consequently, such a situation of self-learning is difficult to simulate in the classroom, and I do not know of any research project in mathematics education that would study the conditions of learning for life-long learning in mathematics. We might benefit, however, from attempts to lay a lot of agency for their learning of mathematics in the hands of the students, for example, by providing them with the learning goals and necessary materials for two school years and trusting them to organise their learning themselves during that period of time (Karner, 2019).

Eventually, although it appears convincing that integration into specific mathematical practices would be a precondition for political agency as a citizen, it remains unclear how to achieve this. Corresponding research in mathematics education often does not have an explicit focus on civic education and pursues goals that are not necessarily in line with civic education. Apart from that, the impact of these research tradition to the mathematics classroom appears very limited. Consequently, more questions are open than answered. But even if more scholars in mathematics education research would shift their focus on the contribution of mathematics education to integrate students into the role of a critical citizen, this would not mean that such a contribution of mathematics education is possible: On the one hand, it may prove impossible or impractical to get students to acquire the required knowledge and skills. On the other hand, even if the students acquired them, the knowledge and skills might prove insufficient, either because we over-estimated their importance, or because we overlooked other crucial mathematical aspects of citizenship.

Legitimisation through mathematics education for citizenship

Even if students were integrated into mathematical practices and qualified in mathematics to an extend sufficient to understand, use, and question the use of mathematics in socio-political discourses, this does not imply that they would use these abilities. Another component is motivational and circles around the question of who has the ability, the right and the obligation to use and check mathematics in these situations. In sociology, school is shown to have the function to legitimise institutions, people and practices in such ways (Kollosche, 2018). Here, we will see the clearest contradictions between school reality and the requirements of civic education.

Scholars from different countries have reported that, in the public, mathematics is usually conceived of as a discipline that can be applied anywhere and that will always lead to unambiguous answers, if it is handled correctly (Borba & Skovsmose, 1997; Davis & Hersh, 1980; Porter, 1996; Ullmann, 2008). While mathematics may be close to this ideal of objectivity when it comes to what is called “pure” mathematics, it is evident to anybody who has seriously applied mathematics that the application of mathematics to real-world problems is far from objective. By this I mean that many application problems do not have an unambiguous answer, and usually different mathematisations can be conceived and found legitimate. We may still try to regain objectivity by seeking a more detailed definition of our original problem (as the scientist does by controlling the variables of experiments), but this cannot totally prevent ambiguity. Meyerhöfer (2013) presented the mathematical modelling of an initiative in transportation planning, which had been repeated and altered until the outcomes matched the political interests of the city government, as a nice example for discussion in the classroom.

When we ask ourselves where this conception of an infallible mathematics comes from, we have to look at the discipline of mathematics first. Indeed, for centuries, mathematics had been praised by academia for its infallibility, and mathematicians might have enjoyed and identified with this prestigious role of mathematics (Davis & Hersh, 1980, 1986). Today, we have to question not only the objectivity of applications of mathematics, but that of “pure” mathematics as well. Yet, it is questionable in how far contemporary experts in mathematics have a well-informed understanding of the limited objectivity of mathematics. Courses in the philosophy of mathematics do usually not constitute a part of university studies in mathematics, and contemporary statements of mathematicians often leave a rather un-informed impression. In his dissertation, Ullmann (2008) discusses applications of mathematics to engineering, to quantitative research, and to politics, and shows how the myth of infallibility has been reproduced there. Also, the persistence of this myth explains why mathematics is used to present fake-news in a more factual light (Hauge, 2019).

Returning to education, we also have evidence that such an understanding of mathematics is reproduced in schools. Dowling (1998) gave us empirical evidence through his analysis of English mathematics textbooks. Ullmann (2008) made a similar point in his analysis of German textbooks. Already in 1979, Keitel speculated about the conception that students can develop of the role of mathematics in reality on the basis of their experiences in the classroom:

They learn from an early age that all problems – at least all those they get to know in applying arithmetic – can be solved with its help and without ambiguity. In the case of problems that cannot be solved, it is not mathematics that fails, but the student; an incorrect solution only means that you have miscalculated. This is how the argumentative stringency of calculations of any kind arises, the persuasive power of numbers, regardless of how they are arrived at, which has its effect wherever someone who has usually learnt maths for longer wants to convince those who have usually learnt less maths of the logic and necessity of an intention or measure. (p. 257, my translation)

Keitel connects her speculations to the power of the use of mathematics in public debates: Assuming that there must be one and only one mathematical answer to a socio-political problem, we do not face any questions of the applicability of mathematics, of the choice of specific mathematical models, or of the reformatting of the initial real-world problem by such a choice of mathematics. Such a conception of mathematics limits our political agency and stands in conflict with the idea of citizenship.

Yet, even if students gained the insight that applications of mathematics can be questioned, this does not mean that they themselves would feel ready to do so. Different studies show that many students find mathematics to be too difficult for them to handle (e.g., Kislenko et al., 2007; Kollosche, 2017). Students' confidence into their mathematical ability appears to decrease during their school career (Di Martino & Zan, 2011). The same appears to be true for the students' willingness to question applications in mathematics. For example, think of Baruk's (1985) famous study using absurd problems such as the age-of-the-captain problem ("On a boat there are 26 sheep and 6 goats. How old is the captain?"), where Baruk shows that students learned in school to provide mathematical answers despite their reluctance to do so in earlier school years! It seems fair to say that many students learn that mathematics is too difficult to understand for them. This experience is amplified by the central role that mathematics plays in education, for example in its assessment and gatekeeper function (Biesta, 2009).

In summary, we can see that contemporary mathematics education seems to contradict the goals of civic education: Neither does it present mathematics as a discipline whose applications can be biased and should be questioned, nor does it equip learners with an identity of competent applicants and critiques of mathematics. However, there is no structural necessity to make this experience when learning mathematics, as alternative organisations of mathematics education show (e.g., Andersson et al., 2015; Steflitsch, 2023). Conceptualisations of mathematics education such as Skovsmose's (1994) concept of Critical Mathematics Education aim at rethinking mathematics education as a contribution to civic education. Examples from implementing such philosophies (e.g., Gutstein, 2003), although not uncontested (e.g., Brantlinger, 2011), provide ideas of how this can be achieved.

Conclusion

In the beginning of this paper, we learnt that democracy requires its citizens to assume an active role in obtaining it. Citizenship, among other things, requires citizens to develop political agency. As socio-political discourses in Modernity are highly formatted by mathematics, education in mathematics plays a crucial role in education or citizenship.

However, the practice of mathematics appears to contradict these goals of civic education. Mathematics education is highly focussed on learning mathematical knowledge and techniques. Foci on mathematical thinking, including mathematical reasoning and mathematical modelling, are less prominent in the mathematics classroom; and where they are, they are often not directed at developing citizenship. Further, mathematics education facilitates an understanding of mathematics and of one's mathematical potential that hinders political agency rather than supporting it. In addition, mathematics education reinforces the popular myth that mathematics can be applied anywhere and yields unambiguous answers, without reflection on the consequences of such applications.

Mathematics education research appears as a part of the problem. It adopts the assumption that mathematics education is mainly about acquiring mathematical knowledge and techniques. When mathematics education research turns to aspects of mathematical thinking, its goals appear to be one-sided: Scholarship on mathematical reasoning sets out to foster skills that are required in mathematics and other sciences, rather than clarifying and studying how such competences contribute to political agency. The same is true for scholarship on mathematical modelling, which tries to educate learners to become virtuous appliers of mathematics rather than to become critical readers of applications of mathematics. Conceptualisations of mathematics education which focus on establishing a different understanding of the socio-political role of mathematics and of the capabilities of students play a marginalised role in the sense that they have not entered the mainstream of mathematics education scholarship.

Critics are right in pointing out that alternative conceptualisations of mathematics education do not align well with the socio-political circumstances in which mathematics education takes place today (Brantlinger, 2011). This does not only imply that scholarship in mathematics education will have to find good arguments to campaign for changes in the socio-political organisation of mathematics education. It also implies that scholarship in mathematics education will have to find ways to think holistically about the goals of mathematics education. The result could be a robust theory of the goals of mathematics education, in which Critical Mathematics Education would constitute only a part, but which would reserve a central place for education for citizenship. Note that, among the 216 entries in Lerman's (2020) *Encyclopedia of Mathematics Education*, there is no single entry on the aims and goals of mathematics education! The fact that there is no discourse on such a teleology of mathematics education in our scholarship apart from a few, hardly recognised examples (e.g., Ernest, 2000; Heymann, 1996/2003), means that those defining the goals of mathematics education – be they practitioners, mathematicians, politicians, or others – have to do so on academically shaky grounds and shows severe shortcomings of contemporary scholarship in mathematics education.

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