THE FORMAL PRESENTATION LANGUAGE OF MATHEMATICS AND COMMUNICATION ETHICS[1]

Marshall Gordon

mgordon417@aol.com

> The conventions of language can be formulated differently, but the purpose of language in providing a functioning social description of the world remains constant.
>
> Paul Ernest

Abstract

Mathematics is a formal language where symbols, verbs, and nouns serve to express terms, concepts, and rules that concatenate to definitions, problem-solving procedures and proofs. Taken together they constitute the expository language of mathematics found in journals, textbooks, and demonstrations. As a communication given to informing, there are epistemological and ethical considerations that deserve examination. For in keeping with the commitment to an aesthetic of concision promulgated by tradition, the formal presentation while valuable in furthering the body of knowledge provides the conclusion of an inquiry absent of the language of investigation that informed the decision making so that little if any insight into the creative process is made available. The problematic communication will be explicated with considering mathematics presented to students early in their education as well at university level. The argument is made that the language of investigation - the heuristic actions instrumental for their formulation, ought to accompany the language of formal demonstration so to provide a communication that is in the best interests of students and members of the mathematics profession.

Introduction

The Executive Director of the Mathematics Association of America, Michael Pearson (2019), issued a statement on "The Critical Study of Ethics in Mathematics", written in conjunction with the American Mathematical Society, asserting that "doing mathematics, in and of itself, is a good thing (or at least value-neutral)". But that perspective was not the complete picture. He also recognized the Society of Industrial and Applied Mathematics whose publication, "Mathematics and Ethical Engagement", contained the statement that "one always performs

[1] This paper is reproduced with the permission of the editors of the journal in which it first appears as M. Gordon (2022) The Formal Presentation Language of Mathematics and Communication Ethics. *Journal of Humanistic Mathematics*, Vol. 12, No. 2 (July 2022): pp. 28-48. DOI: 10.5642/jhummath.OOTE2363. Available at: https://scholarship.claremont.edu/jhm/vol12/iss2/5

mathematics in a social and political context, never in value-free isolation" (Part 1). The antithetical perspectives locate the ambiguity and tension regarding mathematics and its ethical role in the mathematics and education communities.

While ethics and mathematics have been recognized as having a non-empty intersection, "Any attempts to raise ethical issues with regard to pure mathematics are seen as possibly tainting or lowering the subject from its elevated state of purity" (Ernest, 2020, p. 17). But, as pure mathematics research is "almost devoid of ethical context, then it becomes all the more essential [as mathematicians] to heed our general ethical obligation as citizens, teachers and colleagues" (Hersh quoted in Pearson, 2019). The point of concern in this paper is that in keeping with the commitment to an aesthetic of concision promulgated by tradition, the formal presentation while valuable as a communication in furthering the body of knowledge provides the conclusion of an inquiry which is absent of the language of investigation that informed the decision making so that little if any insight into the creative process is made available. This would suggest that an ethical imperative faces the presenter of mathematics with regard to establishing a valuable and valued communication. After all, "Communication's powers . . . to repress and to inspire , . .   to oppress and to comfort, to deceive and to enlighten, . . . [locates] the direct link between communication and ethics" (Makau, 2009, p.1).

Many educators recognize their ethical obligation to create an inquiry environment in their mathematics classroom in an effort to support student understanding. Yet the accepted form of demonstration and the procedures that tend to constitute the presentation of mathematics can be seen to begin with considering that mathematics represents a formal language (cf. Hestenes and Sobczyk, 1984; Morgan, 1996; Silver, 2017; Wilkinson, 1987). Symbols, verbs, and nouns serve to express rules, concepts, and terms that concatenate to definitions, problem-solving procedures and proofs. Taken together they constitute the expository language of mathematics found in mathematics journals, textbooks, and demonstrations. Its expression reflects the aesthetic of concision valued by the mathematics community and reflects that community's commitment to demonstrating mathematics' "austere beauty". The commitment to presenting mathematics absent of the inventive signs of human exploration began with Plato (cf. Gordon, 2019) and gained practical support in past centuries from the very experience of producing mathematics. Many procedures come from a time when writing materials were rare and costly and writing was laborious. Clearly the goal would be to find a most efficient means so to save materials and energy, including parchment and goose quills, so the less that was needed to be written the better. (Consider, e.g., the standard procedures for adding a column of numbers or multiplying multi-digit numbers, and how counter-intuitive they are.) Further support for promoting brevity was the ever-expanding accretion of knowledge. Given the continuing expansion of content and the limited pages of a book along with the limited number of classroom hours, the need for the efficient transmission of procedures and demonstrations in the textbook and the classroom often meant there is limited opportunity for explanation other than to acknowledge that what was presented solves the problem.

The fact that the language presented in mathematics textbooks and as a consequence in many classrooms reflects a commitment to conciseness brings into question the ethical nature of the formal communication. For in the absence of considering the inventive decision-making language that led to its being established, students could well experience confusion and tension. The research literature locates the uniqueness of mathematics amongst all the subjects studied with regard to promoting stress and even phobic responses (cf. Boaler, 2008; Burns, 1978; Hersh and John-Steiner, 2011; Maloney and Beilock, 2011). The problematic epistemological and ethical reality is that "In presenting mathematics [as a finished and polished product], students are provided with mathematical information about concepts, proofs, techniques and skills, but the processes which created this information are hidden. The lack of awareness of these creative processes makes it difficult for students to experience mathematics as personally meaningful and misrepresents the nature of mathematics itself" [Crawford, et al.,1998, p. 466]. And of course, makes it difficult for students to produce work of their own.

The foundational concerns from communication ethics helps sharpen the focus. "In short, communication ethics concerns the discernment of the good, seeking to balance the competing values, needs, and wants of multiple constituencies". . . . That is, communication ethics looks not merely at individual agency and intersubjective processes but also at institutional norms, structural arrangements, and systematic patterns" (Lipari, 2017). Looking through the lens of communication ethics to explicate the presenting of mathematics raises considerations of power, authenticity, integrity, truthfulness and truth. For example, how students can come to feel more powerful in their exploratory efforts if the language for that activity is given little acknowledgement. It is not that the presented material doesn't demonstrate what was to be demonstrated. It is the integrity of the communication that is in question as the formal language circumscribes the language needed to understand how the formal procedure or argument was arrived at. That is, the "Deductivist style hides the struggle, hides the adventure. The whole story vanishes" (Lakatos, 1976, p. 151). And it is that exposition of the general strategies that informed and shaped the formulation that makes evident what doing mathematics is about.

The value of including both problem-solving procedures and problem-clarifying strategies (heuristics) so as to provide insight into the process and product of engaging mathematics was recently shared in the particular instance by the mathematician Curtis T. McMullen who, in commenting on Dennis Sullivan his doctoral advisor and the recipient of the 2022 Abel Prize, noted "The tools that he used, and even more so the analogies that he put to the fore, have been guiding the field [complex dynamical systems] ever since" (Chang, 2022)

The paper to follow will look at the formal presentation of mathematical procedures, definitions, and proofs as they are traditionally offered in textbooks, classrooms and journals beyond the first few years of students' education and reconsider them in the language of heuristics. The point to be made is that the aesthetic of *presenting* mathematics does not acknowledge the informal language associated with the aesthetic of *doing* mathematics, and that is a profound problem in mathematics education. The prevailing cultural belief that "The more you have to put into an argument, in terms of prerequisite knowledge, the more elegance the argument loses" (Dreyfus and Eisenberg, 1986, p. 3) recognizes a commitment to other than the recipient of the communication. As such, the formal demonstration format while valuable for some readers and the furthering of mathematical knowledge would naturally impact the teaching and learning of mathematics, including how students feel about their capacity as mathematical problem solvers, and makes explicit the ethical dimension of the enterprise. After all, “Mathematics as presented” is fundamentally different from the “mathematics in the making”, as Polya noted. With the demonstration absent of its formative development, the authenticity of the communication is bifurcated – it is at once authentic to the canon but inauthentic in its communicative value for the student. Schoenfeld, writing about his experience teaching mathematics and that of the profession, shared that “In presenting a polished solution [‘a part of our professionalism’], we often obscure the processes that yielded it, thus giving the impression that things should be easy for people who study the subject matter. In consequence, the give-and-take of real problem solving . . . are all hidden from students. Yet these are the processes that must be brought out into the open” (1989, p. 200). That they aren’t included in the traditional presentation of mathematics locates the foundational epistemological and ethical communication problem for mathematics educators and students.

The Language Problem by Example

Polya demonstrated how essential heuristics is to the development of mathematics, and thought of it as the “the study of means and methods of problem solving” (1962, p. vi). He distinguished “mathematics in the making” and “mathematics as presented”, but with the presentation not acknowledging the “making”, mathematics is seen as having a disconnected “front and a back”, with the “front” the presented formal demonstration and the “back” the investigative thinking that led to establishing the mathematics omitted from public view (Hersh, 1991). Yet it is how mathematics texts traditionally present mathematics procedures, definitions, and proofs. “The outcome may be elegant texts . . ., but they also generate learning obstacles through [the] reformulation” (Ernest, 2008, p. 67). The consequent fragmentation of knowing and doing could well lead students to make ultimately unwarranted judgements about themselves and their capacity to be successful in fields requiring the engagement of mathematics. That is, in the absence of their gaining understanding, students can’t trust themselves to think constructively nor trust the textbook to provide insight into how to solve problems.

Regarding Definitions

Commitment to an aesthetic of concision comes at a cost with regard to mathematics presentations in all its constituent elements. As regards definitions, while Bertrand Russell held that definitions were value-free, by which he meant in contrast to statements in the logic calculus with associated true-or-false values, definitions are not value-free in terms of the psychological and epistemological weight they carry in that they represent the essential material for developing a framework for the body of knowledge (Gordon, 2011).

In this spirit it would seem reasonable to ask, for example, is there no reason to consider why the definition of the slope of a line is stated as the "change in y over the change in x" and not the "change in x over the change in y"? Were the reciprocal slope expression to be included as part of the classroom discussion a valuable learning moment would be established as the consequence of students' investigation. It would provide students the opportunity to determine what would be the more valuable definition. Here students would have opportunity to engage in an exploration likely involving the heuristics of *tinkering*, *visualizing*, and *generalizing from the particular*, toward coming to a richer and more personal appreciative understanding of the definition in practice along with their growing capacity for productive agency, which is a most critical developmental need for democracy. With the passing over of the informal investigation, students are being informed as to what is, not what makes sense as the result of their active engagement and informed decision making. The disregard is compounded in the particular context with the exclusion of considering why *m* would be used to represent the slope of a line and not the more suggestive *s*, as the latter would logically seem more appropriate. Were the aesthetic of presenting mathematics to include the commitment to supporting and informing students' burgeoning intuition, reflective judgement, and emotional resilience, the communication that would constitute their educational experience would take a more responsive and responsible epistemological and ethical turn. In the absence of such affirming considerations, students who reflect on their experience are left to wonder how the definition came to be, and why their reasonable concerns are not part of the conversation. In that unaesthetic context, alienation associated with the study of mathematics would seem to be being promoted (cf. Walshaw, 2014; Ernest, 2018).

Presenting definitions absent of consideration of the critical development that inspired their coming into being tacitly suggests that mathematics exists other than from the creative energies of human effort. Consider a mathematical group. Here students are traditionally presented with the definition and some examples to make clear its details. In this way the reader misses the opportunity for a deeper understanding, as "Focusing on the heuristics that gradually have led to its formation and refinement . . . displays paradigmatic features of the core of problem solving" (Ippoliti, 2020, p. 1). In his richly developed paper, "Manufacturing a Mathematical Group: A Study in Heuristics", Ippoliti "examine[s] the seminal idea resulting from Lagrange's heuristics and how Cauchy, Galois and Cayley develop it" (2020, p.1). In that article, he makes eminently clear how the foundational development of a mathematical group began with Lagrange drawing upon the heuristics of *look for similarities*, *change of representation*, *generalize from particulars*, and *reason by analogy*, all of which go unconsidered in the definition's presentation. In that absence valuable means for constructing mathematical concepts are omitted and raise questions regarding the ethics of the communication. Were heuristic considerations common to the mathematics textbook and journal presentation and by extension the classroom conversation, there would be greater opportunity to gain a more realistic and appreciative understanding of how the body of mathematics comes to be.

Regarding Procedures

Going back to at least the middle of the 19th century (cf. Davies, 1850), “invert and multiply” has been presented to students for solving the problem of dividing by a fraction. Revisiting the traditional procedure from a heuristic perspective is instructive. In its absence, students are provided with a technique but with little if any understanding of the investigatory considerations regarding how it may have been discovered. The heuristic of *making the problem simpler,* which Polya acknowledged in his *How to Solve It* (1945) and Devlin shared is “how we do mathematics”, provides valuable means for gaining light and happiness. With the change of focus to the question of how the problem can be made easier, students have a potentially promising place to begin rather than focusing on their not knowing how to engage the problem. Dividing by a fraction can be quite challenging even with the digits being elementary counting numbers. Were the focus on how to *make the problem simpler*, discussion can discover the value in changing the denominator to 1, for then the transformed numerator would be the answer. Two approaches presented themselves to students: add/subtract the numerator and denominator by some number or multiply the numerator and denominator by the reciprocal of the present denominator. Performing the first approach the practitioners see the approach doesn’t work. In welcome contrast, with multiplying both the numerator and denominator by the reciprocal of the denominator so to create a denominator of 1, plausible reasoning secures the answer. What gets confirmed with this student engagement is that focus on *making the problem simpler* was critically important, and that the productive resolution was of their doing. More completely, they appreciate their developing educated intuition and naturally growing confidence in engaging mathematics. In this way they come to understand that their mathematics experience is not exclusively determined by whether they have or haven’t memorized a procedure, but the consequence of their thoughtful inquiry supported by a language for investigation. In this way they gain power in being able to engage mathematics productively. With the traditional presentation format of demonstrating procedures rather than promoting heuristic inquiry, students remain passive observers even at advanced levels of mathematics study.

The Method of Partial Fractions is a problem-solving procedure used to solve a class of integral problems in the study of calculus. While students may reasonably try integrating certain expressions by the methods they learned prior - by substitution or by parts, neither are successful with some rational functions. What to do? The textbook provides students with the procedure, and the accompanying presentation of problem solutions demonstrates its value. However, the student is not making decisions with regard to engaging the new mathematics. Were the student to have learned in their earlier mathematics education when faced with a difficult problem that it is valuable to *make the problem simpler* along with another general strategy of *take things apart*, they might well be able to make inroads themselves not only in this situation but in other mathematically such challenging moments (cf. Gordon, 2013 and 2021). But with heuristic considerations often the hidden constant in formal textbook presentations, students would tend to be at a loss for direction and the authority of the textbook demonstration of the particular method would naturally be of limited educational value.

Regarding Proofs

The traditional formal demonstration of mathematical proofs also shares the same foundational language problem of not including the heuristic decisions that were critical to informing and shaping the final argument. While a proof serves many purposes (de Villiers, 1990), at bottom it is a communication presented in good faith by the practitioners of the discipline given to the promotion of mathematics as a developing body of knowledge and the reader's understanding of what is the foundational element of the discipline. From that vantage point, "Although it is easy to adopt an ethically neutral approach when discussing organizations and communication, this is simply not an option. One ought to place ethics as a first principle of communication" (Seeger, 1997, p. xii). Toward its more complete representation, two proofs will be considered along with a framework that demonstrates how heuristics can accompany formal mathematical demonstrations so as to be both epistemologically and ethically responsible.

With the cultural commitment to the presentation of deductive reasoning exclusive of recognizing intuition and the language of heuristic investigation that gave it form, plausible reasoning regarding how a proof could unfold is not part of the traditional demonstration. Such a limited framework makes evident the problematic nature of the communication to readers with respect to gaining understanding and, as a social practice naturally raises ethical concern as well. After all, ". . . it is the intuitive bridging of the gaps in logic [in a proof] that forms the essential component of the idea and its implications" (Hanna, 1989, p. 23). So it could well continue that students would naturally and logically draw the inference that it is their shortcomings that is the determinant of the difficulties they are having in trying to analyze and generate demonstrations. Yet "The nature of that tension is anything but new. We have known it in the philosophy of science in the form of the context of discovery versus context of justification divide. A justification is preferably seen as something independent from the discovery process. The processes that led to the proof are of no importance" (Van Bendegem, 2014, p. 267). Such an epistemological disconnect naturally and logically raises ethical concern as "The formal-logic picture of proof is not a truthful picture of real-life mathematical proofs" (Hersh, 1993, p. 391). With the essential mental actions that shape the creative engagement in establishing a mathematics proof left unconsidered, it makes sense that in a course of undergraduates transitioning to proof. "All of the students said they had relied on memorizing proofs because they had not understood what a proof is nor how to write one" (Moore, 1994, p. 264).

The problematic nature of proof demonstrations as a communication has been recognized not only in research regarding student practice but by mathematicians as well (Thurston, 1994), and can be realized in what is considered by the mathematics community as a "good" proof. At present, the thinking is that a "good" proof is one which demonstrates *that* a proof is valid and *why* it is so (Hersh, 1993; Rav, 1999; Byers, 2007), as will be seen in the two proofs to follow. Yet what is not included is the heuristics that share *how* the proof came to be constructed, and in that omission an ethical and epistemological dilemma resides.

The two proofs that follow will be revisited through a heuristic lens. The concluding discussion will present a framework within which the formal proof demonstration can be accompanied by heuristic analysis without disfiguring the traditional presentation model of mathematics.

Euclid's Proof of the Sum of the Angles of a Plane Triangle

As the reader well knows, Euclid's demonstration of the sum of the angles of a plane triangle required establishing a 5^{th} postulate, his Parallel Postulate. Indeed, the first step of the argument, "Draw a line through a vertex parallel . . ." contains in essence the entire proof. The proof provides a convincing argument *that* the sum of the angles is 180^{o} and an explanation *why* as a consequence of parallel lines, but *how* the argument came to exist is not made explicit. In that absence, the reader is left with imagining inspiration/intuition as providing the defining moment. While that is likely true, it doesn't further understanding as the heuristic decisions that enabled the argument are kept in the shadow.

"It is thought that Euclid must have studied in Plato's (430 B.C.E.–349 B.C.E.) Academy in Athens, for it is unlikely that there would have been another place where he could have learned the geometry of Eudoxus and Theaetetus on which the *Elements* is based (Krantz, 2007, p. 14). As drawing diagrams and trying to discern what if any mathematical relationships could be established was a common practice, one could imagine that he along with others in an effort directed at determining the sum of the angles of a plane triangle would *make the problem simpler* by likely choosing the heuristic of *taking things apart*, and with compass and straight edge arrange replications of the angles of a triangle to find a straight angle. The potential proof maker(s) may well have determined other triangle angle sums and would likely have used the heuristic of *generalizing from the particular* to provide the impetus to try to prove that every plane triangle angle sum was 180^{o}. The problem they faced was how to demonstrate that the straight angle of 180^{o} had the same sum measure as the non-linear triangle angles. Euclid's determined imaginative effort informed by *tinkering* would be rewarded by *visualizing* the definitive connection: the triangle angles and the straight-line angles in conjunction with parallel lines. And that creative engagement gave birth to the foundational Parallel Postulate (Proposition 32 in Book 1). But with all signs of the heuristic engagement, such as those above, absent from textbook presentations a more informed understanding remains at a distance from students, suggesting confusion could well be the natural emotion and memorization the logical solution for many as the initial step is presented as if it were clear it was the natural place to begin.

Oresme's Proof of the Divergence of the Harmonic Series

To prove the harmonic series diverged Nicolas Oresme introduced a second infinite series whose sum can be recognized as less than the harmonic series (see Figure 1):

Figure 1

$HS = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \ldots$, and
$HS \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \ldots$.

In the second series, terms of the harmonic series have been replaced by terms with denominators of lesser or equal quantity, yet establishing a series that clearly diverges as terms have been collapsed into groupings having a sum of 1/2. This elegant formulation by Oresme has it that the 3rd and 4th terms sum to ½, 5th through 8th terms sum to ½, 9th through 16th terms sum to ½, . . ., with the constant partial sums constituted by 2^{n-2} terms of the form $\frac{1}{2^{n-1}}$, $n \in \mathbf{Z}^{+} \geq 3$ which establishes a series of infinite halves that clearly increases without bound.

While his proof is part of the well-accepted body of mathematical knowledge, and seen as “elegant” and “beautiful”, it has left some students and mathematicians uncomfortable for lack of being able to discuss how one would have known to do what Oresme did. The fact that the method of proof is indirect tacitly communicates to the experienced reader that going at the problem directly was apparently found to be too difficult (and remains so). While mathematicians may realize that, it is not clear that students who have considerably less experience in proving would have that awareness. Oresme’s choice of approach would have to be as commentary accompanying the proof; in its going unmentioned the proof can be seen to in effect start in the middle. When one finds Oresme’s delightful proof statement in print there doesn’t seem to be any mention the driving impetus was to *make the original problem simpler* – a fundamental strategy for dealing with challenging problems, nor any mention of the heuristics (problem-clarifying strategies) of *tinkering* and *taking things apart* that would seem to have led Oresme to formulate the proof as he had.

That is, while his demonstration made clear that the theorem is valid and why it is so, its inventive formulation remains outside the product presentation as the consequence of the language of heuristics being at a distance from the standard language of mathematics argument. That heuristics constitute in effect a meta-language distinct and culturally distant from the language of formal mathematics is made explicit by a mathematician, author of a proof by contradiction that the harmonic series diverges, who expressed wonder with regard to Oresme’s proof, asking “How would one come up with the idea of grouping more and more terms together?” (https://web.williams.edu/Mathematics/lg5/harmonic.pdf). Such expression makes clear that “The language of heuristics is at a distance from contemporary mathematicians as a consequence of not having been educated with regard to this body of instrumental knowledge” (Polya, 1962, p. viii). Yet Halliday (1975) found children’s language to include a “heuristic function” which he described as the “language that is used to explore, learn and discover”. That is, it is natural to the thinking that we all do to inquire productively, but yet it remains outside the communication of formal mathematics.

This is to say, heuristics need not be seen as a competitor to nor be kept separate from proofs. The distinction between “demonstrative reasoning” and “plausible reasoning” (the “front” and the “back”) need not be seen as problematic. After all, “they don’t contradict each other; on the contrary, they complete each other” (Polya, 1954, p. vii). With including the explanatory *how* of the proof argument, the communication is made more whole and the mathematics educator’s ethical imperative to promote student understanding is secured as the generative thinking that informed and shaped its formulation is made present. And it can be done in a framework sensitive to the readership and the traditional model of proof demonstration. That is, for the mathematics community, the essential heuristic thinking could be located after the proof presentation. In this way the proof can be appreciated absent of any scaffolding, of being annotated, yet with the opportunity to gain a more complete understanding still available. For students, the heuristic description could well be more valuable were it to precede textbook demonstrations, as students would have the opportunity to secure an understanding of the critical underlying thinking that would follow. Moreover, the framework’s epistemological value is supported by the underlying ethical commitment that is acknowledged with the more complete communication.

In Conclusion

The presentation of mathematics is a complicated affair. The mathematics-maker acts authentically with developing definitions, procedures, and proofs that inform and can inspire the mathematics community. Yet at the same time the formal presentation language is not authentic absent of the underlying heuristic considerations that would serve to provide a rationale for the content’s construction. It is a profound problem in mathematics education, as so many students (read people) have difficulty with mathematics, though we are born with the capacity to recognize pattern, to generalize, and provide justification - with the capacities to do mathematics. And yet the experience of so many students is not one associated with developing understanding, with becoming a more competent mathematical thinker. It could be that some who teach mathematics may well have difficulty in talking about mathematics so as to enable students to be able to engage it intelligently as for the presenters it comes so naturally it is difficult to explain. Many of us who have studied mathematics have had the experience of the presenter who was not aware that students were having a difficult time following the presentation, as for them the logic of their communication was very apparent. There is another group which may well represent the more common classroom experience students have. The joint American Mathematics Society and Mathematical Association of America report, *The Mathematics Education of Teachers* II, shared the concern that “For many prospective teachers, learning mathematics has meant *only* learning its procedures and, they may, in fact, have been rewarded with high grades in mathematics for their fluency in using procedures” (AMS, p.11; emphasis in the original). And because they were good at demonstrating procedures, it makes sense that procedural learning would be at the foundation of their mathematics teaching. The authors go on to point out that the students they taught came to problematic beliefs about mathematics and learning mathematics and, were they not able to demonstrate those procedures, about themselves as students of mathematics. Research findings and classroom discussions make that clear. For while gaining understanding often requires a “messy” inquiry process, the polished presented product of the investigative experience is a poor representation of the experience yet it is what tends to be communicated. Many students have difficulty in school with regard to their mathematics study, and the only constant this author can see is the formal demonstration model that populates standard textbooks. This is to say, epistemologically and ethically, the relation between student and teacher is a fundamental relation, whether in or out of school, The nature of the communication is where we in the mathematics community must put dedicated energy. This most especially needed in a democracy where everyone is counted on to participate to the best of their abilities. For that to happen requires resilience, patience, and perserverance to stay with a challenging problem.

Toward promoting and supporting those capacities in student effort-making a dual-presentation framework could also be brought to the Answer section of the traditional mathematics textbook where the student finds out if they are right or wrong. Most generally, if the student’s answer isn’t the same as what is in the Answer section they can return to investigate the problem further and may discover where they made some error in judgement or practice that sent them on a “wild goose chase” (as Schoenfeld calls misguided investigations), and actually reach a new conclusion to be checked.   But suggestions regarding how to engage the problem for those who haven’t arrived at an answer or for those whose answer is not the same as in the back or have no idea as how to reengage the problem are traditionally absent from the Answer section. Were there two sections - Questioning and Answer sections, where in the former the student could find instrumental heuristic hints as how they might “ask a (good) question” that would support their making a further effort in addition to having available the Answer section, there would be more opportunity to move beyond coming to know if one is “right or wrong”. A problem-clarifying strategy would be available for students to reengage the problem toward more satisfying resolution. In this way there would be further established the writers of mathematics textbooks ethical commitment to support students’ mathematical intuition and dedication to reason to a valued and valid conclusion.

The generative heuristic insights essential for establishing mathematical definitions, procedures, and proofs can be incorporated without diminishing the traditional presentation model. With the inclusion of heuristic terminology in mathematics classrooms, textbooks, and journals, a universal language that communicates across the boundaries of mathematical areas of investigation and development would be made available. With this encompassing quality, mathematics can be more completely appreciated by professional practitioners as well those who are relatively new to its practice for the depth of communicative understanding it would provide. The more cohesive, coherent, and honest representation of the inquiry experience means the ethical dimension of the presentation of mathematics as a communication would be secured along with a more complete instrumental understanding of the development of the body of mathematical knowledge. In this way the opportunity for mathematics becoming more available to a greater and more appreciative participatory audience comes to be.

References

American Mathematical Society. (2012). The mathematics education of teachers II. *Conference Board of the Mathematical Sciences*, Vol. 17, authors.

Boaler, J. (2008). *What's Math Got to Do With It?* New York: Penguin Books.

Burns, M. (1975). *The I Hate Mathematics Book*. California: Yolla Bolly Press.

Byers, W. (2007). *How Mathematicians Think*. Princeton: Princeton University Press.

Chang, K. (2022) Abel Prize for 2022 Goes to New York Mathematician. *New York Times*. 23 March 2022

Crawford, K., Gordon, S., Nicholas, J., and Prosser, M. (1998). Qualitatively different experiences of learning mathematics at university. *Learning and Instruction*, **8**(5), 455–468.

Davies, C. (1850). *Elementary Algebra: Embracing the First Principles of the Science*. New York: A. S. Barnes & Company.

De Villiers, M. (1990). The role and function of proof in mathematics, *Pythagoras*, **24**, 17-24.

Dreyfus, T. and Eisenberg, T. (1986). On the aesthetics of mathematical thinking. *For the Learning of Mathematics*, 6(1), 2-10.

Ernest, P. (2008). *Opening the mathematics text: What does it say*? In de Freitas, E. and Nolan, K., (Eds.), Opening the Research Text: Critical Insights and In(ter)ventions into Mathematics Education. Dordrecht. Springer, 65–80.

Ernest, P. (2018). The ethics of mathematics: Is mathematics harmful? In P. Ernest (Ed.), *The Philosophy of Mathematics Education Today*. Switzerland: Springer.

Ernest, P. (2020). The ethics of mathematical practice: From resistance to realisation and responsibility. In B. Sriramen, Ed., *Handbook of the History and Philosophy of Mathematical Practice.* Switzerland: Springer.

Euclid. (1956). *Elements.* New York: Dover.

Gordon, M. (2011). Mathematical habits of mind: Promoting students' thoughtful considerations. *Journal of Curriculum Studies*, Vol. 43, No. 4, 457-470.

Gordon, M. (2013). The mathematics of fountain design: a multiple-centres activity. *Teaching Mathematics and its Applications*, 32(1), 19-27. Oxford University Press. (Also appears in *The Best Writing on Mathematics – 2014*, M. Pitici (Ed.), Princeton University Press.)

Gordon, M. (2019). Identity, culture, and pedagogical challenge: The presentation of mathematics. *Philosophy of Mathematics Education Journal*, No. 35, December.

Gordon, M. (2021). Teaching mathematics - Heuristics can and ought to lead the way. *Journal of Humanistic Mathematics*, 11(2), 392-404.

Halliday, M. A. K. (1975). *Learning How to Mean*. London: Edward Arnold.

Hersh, R. & John-Steiner, V. (2011). *Loving and Hating Mathematics*. Princeton: Princeton University Press.

Hersh, R. (1991). Mathematics has a front and a back. *Synthese*, **88** (2), 127-133.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, **24**, 389-399.

Hestenes, D. and Sobczyk, G. (1984). *Clifford Algebra to Geometric Calculus: A Unified Language of Mathematics and Physics*. Reidel Publishing.

Ippoliti, E. (2020). Manufacturing a mathematical group: A study in heuristics. *Topoi – An International Review of Philosophy* 39, 963-971. Published online 19 February 2018, https://doi.org/10.1007/s11245-018-9549-1

Krantz, S. G. (2007). The history and concept of mathematical proof. *American Institute of Mathematics*, http://www.math.wustl.edu/~sk/eolss.pdf.

Lakatos, I. (1976). *Proofs and Refutations: The Logic of Mathematical Discovery*. Cambridge University Press.

Lipari, L. (2017). Communication Ethics. *Oxford Research Encyclopedia of Communication.* Retrieved 1 Dec. 2024, from https://oxfordre.com/communication/view/10.1093/acrefore/9780190228613.001.0001/acrefore-9780190228613-e-58.

Makau, J. M. (2009).Ethical and unethical communication. *21st Century Communication: A Reference Handbook*, W. F. Eadie (Ed.). Thousand Oaks, CA: Sage Publications, 433-44.

Maloney, E. A. & Beilock, S. I. (2013). Math anxiety: Who has it, why it develops, and how to guard against it. In M. Pitici (Ed.), *Best Writing on Mathematics*. Princeton: Princeton University Press.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics,* **27**, 49-266.

Morgan, C. (1996) 'The language of mathematics: Towards a critical analysis of mathematics texts.' *For the Learning of Mathematics*, **16**(3), 2-10.

Pearson, M. (2019). The critical study of ethics in mathematics, Parts 1-3. Retrieved on 1 Sept. 2024 from https://www.mathvalues.org/masterblog/parts1-3-the-critical-study-of-ethics-in-mathematics.

Polya, G. (1945). *How to Solve it*. Princeton: Princeton University Press.

Polya, G. (1954). *Mathematics and Plausible Reasoning*, two volumes. Princeton: Princeton University Press.

Polya, G. (1962). *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving*, two volumes. New York: John Wiley and Sons.

Rav, Y. (1999). Why do we prove theorems?, *Philosophia Mathematica*, **7**, 5-41.

Schoenfeld, A. H. (1989). What's all the fuss about metacognition? In Schoenfeld, A. H. (Ed.), *Cognitive Science and Mathematics Education*, NJ: Lawrence Erlbaum Associates, 189215.

Seeger, M. (1997). *Ethics and Organizational Communication.* NY: Hampton Press.

Silver, D. S. (2017). The new language of mathematics. *The American Scientist*, 105(6). 364– 371

Thurston, W. P. (1994). On proof and progress in mathematics, *Bull. Amer. Math. Soc*., **30**(2), 161–177.

Van Bendegem, J P. (2014). The impact of the philosophy of mathematical practice on the philosophy of mathematics. In L. Soler, S. Zwart, M. Lynch, & V. Israel-Jost (Eds.). *Science After the Practical Turn in the Philosophy, History, and Social Studies of Science*. New York: Rutledge.

Walshaw, M. (2014). Who can know mathematics? *For the Learning of Mathematics,* 34(2), 2-6.

Wilkinson, L. C. (1981). Teaching the language of mathematics. *Journal of Mathematical Behavior*, 51,167-174.