THE PHILOSOPHY OF MATHEMATICS EDUCATION: STATE OF THE ART

Paul Ernest

University of Exeter, UK
p.ernest@exeter.ac.uk

ABSTRACT

This paper offers a state-of-the-art review in the subspecialist area of the philosophy of mathematics education. It considers various topics including the philosophy of mathematics and mathematics education, philosophy of mathematical practice, proof in mathematics, teacher beliefs as personal philosophies of mathematics (and learning), research methodology, critical mathematics education, ethics, ontology and aesthetics. Within each of these topics some central recent examples of research are discussed critically. Although selective, the end result is a state-of-the-art overview of the philosophy of mathematics education that reveals that it is a flourishing and growing area of research.

Introduction

The Philosophy of Mathematics Education has been a theme of research within mathematics education over four decades. Most *International Congress of Mathematical Education* (ICME) conferences since 1992 have featured topic and discussion groups on this theme.

A number of areas of inquiry have been central to research in the philosophy of mathematics education, such as the philosophy of mathematics and its individual counterpart, personal beliefs about the nature of mathematics, especially teacher and student philosophies. In addition, various further areas of inquiry have opened up to the present time, and which are reflected here, including epistemology – especially proof in warranting mathematical knowledge, research methodology, critical mathematics education, ethics, ontology and aesthetics.

The Philosophy of Mathematics and mathematics education

The relationship between philosophy of mathematics and mathematics education has been a central topic of research, right from the outset. The works of Imre Lakatos (1976) and others in the ‘maverick tradition’ (Kitcher and Asprey 1985) which look at the social and humanistic side of the philosophy of mathematics (e.g. Davis and Hersh 1980, Hersh 1997) have been central.[1] These have inspired a movement in the teaching and learning of mathematics to try to offer students tasks whose completion parallels what professional mathematicians do in creating new mathematics. These, drawing on the mathematical problem solving and the mathematical investigation movements, have aimed to teach mathematical enquiry as it is conducted by researchers in mathematics, even if only in in miniature.

> Mathematics has two faces; it is the rigorous science of Euclid but it is also something else ... mathematics in the making appears as an experimental, inductive science” (Polya 1957, p. vii).

The idea is that engaging in mathematics as an experimental, inductive science, as opposed to a deductive science, as presented in Euclid’s elements. The question therefore arises, should the teaching and learning of mathematics in school aim to focus on preparing students to act like research mathematicians? A significant response to this question comes from by Weber et al. (2020). They highlight reasons why mathematical practice sometimes should not inform mathematics instruction. The first problem is identifying what strategies and behaviours mathematicians exhibit in solving mathematical problems. Unless these can be specified it is very difficult to plan instruction to encourage ‘mathematician-like’ behaviours. But several studies have shown a great lack of heterogeneity in such behaviours. Burton (2004) studied 70 professional mathematicians and found dramatic differences in the ways that mathematicians solved problems. For example, many mathematicians regularly invoked visual reasoning in problem solving while other mathematicians rarely did so. DeFranco (1996) compared the problem solving behaviour of two groups of eight mathematicians, highly regarded and internationally recognized mathematicians, as opposed ‘ordinary mathematicians’ that had earned a Ph.D in mathematics and published papers in mathematics journals. He found that the “ordinary mathematicians” were not observed engaging in the metacognitive behaviours that have been identified as characteristic of mathematical thinking, by Schoenfeld (1985) and others. Thus, the critique of Weber et al. (2020) is that

[1] The first attempt to draw the lessons of the fallibilist and ‘maverick’ traditions for mathematics education was by A.J. (Sandy) Dawson (1969, 1971).

there is a lack of heterogeneity in mathematical behaviours among mathematicians, so it is not possible to offer a school mathematics curriculum that ‘emulates what mathematicians do’.

A different and deeper has been offered by Stillman et al. (2020). They question the assumption that learning mathematics should mean doing as pure mathematicians at university do, whatever the problems with this perspective may be. They argue that the goals of learning mathematics must include pragmatism and vocationalism. By p*ragmatism* they mean educational goals encompassing Dewey’s (1916, p. 140) notion of “civic efficiency or good citizenship”; the development the capacity for good judgment as an effective member of society. By v*ocationalism* they mean the utilitarian goal of preparation for a life of professional work, as well as managing economic resources efficiently, within a democratic society.

Their important argument is that the activities of pure mathematicians are largely ignored by biologists, engineers, and physicists and in workplace settings. However, the professional modelling practices of applied mathematicians are highly valued. Looking at the preponderance of applied mathematics and modelling practices over pure mathematical activities (20 to 1), in particular in the USA STEM related professions, they argue that learning to use and apply mathematics should be the dominant goal of school mathematics education.

This is a very important conclusion for both democratic and social vocational reasons. Traditionally, both in USA and UK, professional pure mathematicians at universities have had a powerful impact on steering school mathematics towards pure mathematics (Ernest 2014). This goal of mathematics for mathematics-own-sake in education has been detrimental for the majority of students. Their natural interests are more towards using and applying mathematics in their lives, and the world around them, than directed at the internal goal of pure mathematics for its own sake. This argument and finding runs counter to the love of many mathematics education professionals for pure mathematics, but better reflects our responsibilities towards our clients, both students and teachers.

Philosophy of mathematical practice.

In recent years a new movement has emerged, referred to as *the philosophy of mathematical practice.* This looks at the actual social and historical practices of mathematicians from a philosophical perspective. It is well represented in the recent Springer handbook on the history and philosophy of mathematical practice (Sriraman 2021). This contains several chapters treating the living and practical elements of mathematics that are of interest and relevance to mathematics education. For example, the actual role of proofs in mathematical practice, including their formality (and lack of), the values within mathematical proofs, and the dimensions of their acceptance. In addition, chapters on What Mathematicians Do in their processes and Creative Rationality as well as the heuristics of Mathematical Practice, are already reflected in problem solving and investigations movements within mathematics education.

Van Bendegem (2018) provides an overview of the philosophy of mathematical practice and its relations to ethnomathematics, the sociology of mathematics and mathematics education. He acknowledges the key contributions of Ubi D'Ambrosio and Alan Bishop, among others, in extending the study of the spread of mathematical practices to their social historical and educational contexts. He recognises that traditional philosophies of mathematics focused on such recondite questions as what are reliable foundations for the whole of mathematics and is set theory a better or worse candidate than category theory? These are inward looking foci, that turn one's gaze away from the social context in which mathematical activities take place.

From the practice point of view, however, the links with ethnomathematics[2] and mathematics education are evident. For practices are "carried" by people and people have to be educated, that forges the link with education. Further, practices are socially embedded and thus culturally situated, forging the link with ethnomathematics. As he puts it, "education concerns the diachronic dimension of how mathematical knowledge is situated in time, whereas ethnomathematics concerns the synchronic dimension of how mathematical knowledge is situated in space." (Van Bendegem 2018: p. 46). Overall, Van Bendegem's map of philosophy of mathematical practice is a very rich interconnected one which shows many dimensions of mathematical practice and research, including both philosophy and education, are irrevocably entangled. It shows how educational questions matter to philosophy, and how issues from the philosophy of mathematics matter to education.

Proof in mathematics

A central theme in both the philosophy of mathematics and in mathematics education is proof: its purpose, its varying nature and how the skills involved are acquired. The place of proof in mathematics education is a dual one. On the one hand proof is a central element of mathematical content, as well as

[2] Ethnomathematics is a very rich area of research which from its outset, e.g. in D'Ambrosio (1985) has brought up philosophical and social problems concerning mathematics education. It is a vast an area of research, and much of it is anthropological rather than philosophical, and it will not be treated except incidentally in passing in this paper.

associated skills in proving (reading and writing mathematical proofs). This is not a central aspect of research in the philosophy of mathematics education, although important in the study of advanced mathematical thinking. However, the second aspect, the epistemological status of mathematical theorems and the role of proof in warranting them is a philosophical issue, as is the role of proof and its appreciation in contributing to and developing overall and personal philosophies of mathematics. Despite this distinction, it is not easy to distinguish between the philosophical and the mathematical and pedagogical roles of proof in mathematics education.

As Rocha (2019) indicates mathematical proof can have several functions, including verification, explanation, systematization, discovery and communication. All of these functions are important for mathematicians. However, school students usually engage only with the verification function. Furthermore, the circumstances where this contact happens tend to be limited to intuitive results or theorems presented by the teacher, where the verification is not really a question. As a consequence, many students do not understand the point of proving. Thus, there is a real question about including proof in the school curriculum, if only the warranting or verification dimensions are addressed. Rocha suggests that we need to broaden the conception of proof in school mathematics, to include more informal arguments, as well as looking for opportunities to engage students in the different roles of proof. The experiences of proof that school students have are limited to what is offered in the curriculum and also depend on the teacher knowledge, beliefs and their approaches to proof. Providing experiences in identifying incorrect proofs as well as reading and constructing proofs would add to the epistemological empowerment of students, beyond just extending their skill set.

Teacher Beliefs as Personal philosophies of mathematics (and learning)

The investigation of teacher beliefs as personal philosophies of mathematics (and of education) was one of the earliest strands in philosophy of mathematics education research. Ernest (1991) offered a complex model of group ideologies of mathematics education and the individual beliefs of mathematics teachers. This included six primary elements (epistemology, philosophy of mathematics, values, theories of the child and of society, educational aims) and eight secondary elements (the aims of mathematics education, theories of:- school mathematical knowledge, learning mathematics, teaching mathematics, assessment of mathematics learning, resources for mathematics education, mathematical ability, and social diversity in mathematics education). This and similar models have been very influential in research in mathematics teacher beliefs, and in distinguishing between espoused (stated) beliefs and enacted beliefs (behaviours that reflect belief positions, or from which beliefs are inferred). This contrast also mirrors that between the explicit planned curriculum and the taught curriculum. The latter often includes elements of a ‘hidden curriculum’ that influences the learned curriculum, although such elements are not included in the explicit aims. Teacher beliefs as a research topic has also been associated with investigating teacher knowledge, attitudes and beliefs as a cognitive area of inquiry (Liljedahl & Oesterle 2020).

The more philosophical aspect of modern research on mathematics teacher’s beliefs is currently more closely associated with research on teacher identities. Heyd-Metzuyanim (2019) argues that in traditional cognitivist research there is over-reliance on reifications of beliefs. She argues against the conceptualisation of individual’s beliefs as comprising belief systems, which are objects ‘held’ by teachers. From this perspective, beliefs are objectified as self-subsistent and enduring entities which may have persistent impacts on their owner’s classroom practices. The reification of beliefs leads to ‘ontological collapse’ (after Skott 2015), in which a speech utterance (a token) from a research participant becomes transformed into an enduring cognitive object (an abstract type). In contrast Heyd-Metzuyanim is interested in teachers’ endorsed narratives about teaching and learning mathematics, as well as narratives about themselves as teachers and doers of mathematics. These are seen as contributing to a participant’s identity, following Sfard and Prusak’s (2005) definition of identity as a collection of narratives about an individual that are reifying, endorsable and significant (p. 16). This approach offers a powerful tool for exploring individual teacher’s identities, that which makes them unique, albeit located in an ineliminable set of shared cultural matrices.

Unfortunately, in this particular investigation, the opportunity to explore teacher identities in all their rich uniqueness is lost. For the research conceptualized teachers’ identity as drawing on distinct discourses existing in the public sphere. In particular, the research differentiated between the discourse of reform and pedagogical explorations (previously termed progressive) and the discourse of acquisition (traditional knowledge acquisition pedagogy). This submerged the individual identity differences in a traditional binary characterisation of the curriculum and pedagogy. This same critique applies to the albeit more complex characterisation of beliefs in Ernest (1991) described above. Although that theorisation goes beyond a simple binary divide, it nevertheless characterises beliefs as enduring constructs. However, given that the framework is couched in the language of ideologies it is perhaps more defensible than if it was purely expressed in the language of beliefs.

One of the key strengths of moving from a conceptualisation of cognitive beliefs to a more performative, narrative view of identity as evidenced in Heyd-Metzuyanim (2019) is the rejection of the assumption that beliefs are enduring and persistent mental objects. ‘Beliefs’ if one can still call them that, are reconceptualised as fluid, contextually bound and all the while revisable clusters of narratives, affected, like identity, by linguistic and social positioning. Understood this way they are less susceptible to the objectification and essentialising of identity and seen more all the while in a process of becoming and change. There are constancies and recurring themes, but they are not bound into some enduring reification.

Beyond this critique, Chronaki and Kollosche (2019) argue that mathematical identity is entangled in socio-political issues of mathematics education. Therefore, a more a politically sensitive approach to identity research is requires than is widely found in the literature. To accommodate this, they draw on a post-structuralist theory. This is perhaps no surprise as Foucault’s post-structuralist approach to discourse and positioning explicitly revolves around the issues of power, and hence politics. Chronaki and Kollosche draw on the discourse theory of Laclau and Mouffe (2001) to study mathematical identity, not as a fixed but as a contingent meaning-making process that unfolds the political struggles of mathematics education in our contemporary times. This discourse theory proposes a social relational organisation of identity formation as a continuous process of accepting, resisting and reconfiguring notions of self and other in the context of discursive political praxis.
The first

Chronaki and Kollosche’s research methodology is based on the analysis of participants narratives. These begin with searching for contingent nodal points (words), moments and elements in a subject’s interview data and discourse. This first step illuminates which nodes have a temporal privileged status in the narrative and their relations to other nodes. The second step explores how cluster connections between different nodes in an interview text can be mapped and suggests nodes that assume central positions as nodal points. Thus, an articulation such as “I just find maths difficult, I do not understand it that quickly” connects mathematics with such nodes as difficulty, understanding and the pace of learning mathematics. The notion of difficulty can then become a nodal point around which other nodes become organised. Lastly, this discursive architecture is contrasted with alternative discourses as present in the interview text, or even as discussed in mathematics education research.

Chronaki and Kollosche (2019) illustrate their method with the case of ‘Anja’ a 15-year-old female student that expressed a rejection of mathematics. Analysing her responses to a sequence of questions Anja reveals her refusal of mathematics as she states that she ‘never’ wants to attend mathematics, but she is obliged to. The analysis of Anja’s interview focusses on identifying moments, elements, nodal points and their relations to each other to explore how she, in the process of her identity work, strives to articulate partially fixed meanings in her field of discursivity. Among a larger field of terms, represented like a concept map, the researchers found four inter-related nodal points: togetherness, dignity, relevance and bodily activity. These nodal points were not explicit in the discourse but addressed with varying vocabulary, metaphors and stories.

What this case reveals, explored in much greater depth in the paper, is the unique response and identity configuration Anja constructs in response to her experiences of school mathematics. Of course, the researchers interpret Anja's narrative not as an expression of a fixed mathematical identity, but rather as a testimony of her struggles with her everyday social experiences of her school mathematics reality. Furthermore, it is, and can only be, just a sampling of her narratives resulting in a tentative modelling of a small part of a changing mathematical identity.

The researchers go on to question how students experience mathematics, and how mathematics education could be reorganised in the light of their narrative fields. How can participants, who, as in Anja's case, often lack positive terms in personal discourses, pursue their identity work through the articulation of their positions within a broadening discursive field? How can we recognise the potential of emancipatory relations that do not suffocate each student with an ideal 'identity' construct? Although these are very difficult questions to answer, they clearly show that a 'one size fits all' mathematics curriculum that does not accommodate the variety of student mathematical identities is doomed to, at least partial, failure.

Research methodology

The last section implicitly posed the question of the extent to which research methods and methodologies are philosophical. Of themselves research methods are not philosophical, although they rest on philosophical assumptions. Research methodologies, however, should always raise philosophical issues. Sometimes a dominant paradigm of research, such as the scientific research paradigm, becomes so easily and automatically applied that its philosophical underpinnings are not always laid bare, and their intrinsic limitations clarified. However, scientific research paradigm does rest firmly on the philosophy of science and other cognate specialisms in its quest for laws, generalisations and its use of hypotheses and their tests.

The previous inquiry into identity, and mathematical identity, does raise deep philosophical questions. These are ontological questions concerning being and becoming. Furthermore, in a field of study like mathematics education, that is concerned with the social activities and practices of teaching and learning mathematics, a bridge must be found between theory and practice. It is not enough to define mathematical identity without specifying possible approaches for seeing how the definitions and theoretical frameworks cash out in practice. What implications does a theoretical account of mathematical identity have for research investigations of the mathematical identities for students, teachers and us all? Thus, both the implications for research methods and for guiding educational practices, insofar as they can be drawn out of the theories and results, are an important and necessary corollary to philosophically grounded research. Indeed, if philosophical theories had no indications (understood broadly, extending beyond strictly logical implications) for empirical research or practice, it could be argued that they are superfluous and dispensable.

Research methodology is understood as the underlying philosophical foundation that makes explicit the assumptions of any empirical enquiry. It provides the theoretical underpinning and rationale for many of the research processes including question articulation, and choice of research methods; both data gathering, and data analysis. Three main areas of philosophy are foregrounded in research methodology: ontology, epistemology and ethics (Stinson 2020). Stinson, echoing others, claims that these three domains of philosophy are central to all social science and educational research including that undertaken within in mathematics education. Epistemology is central because what is usually at stake is knowledge claims and the warrants for knowledge, as well as individuals' learning, understanding and knowledge. Certainly, this is the case for scientific paradigm research and that conducted in the interpretative paradigm. Somewhat different is the Critical-Theoretic research paradigm where the aim is to change society or some social institution, such as the standardised teaching and learning of mathematics in schools. Here knowledge development, discovery and justification are necessary but not sufficient to achieve the goals of the research, because it is also aimed at contributing to social justice or some cognate aim.

Ontology is also central, for it concerns what there is, that is, being and becoming. Ethics is even more central; it is claimed to be the 'first philosophy of mathematics education' (Ernest 2011). It suffices to say here that what underpins all human interactions and actions is ethics. Ethics encompasses doing good, human flourishing, and all the motives of action including striving for utility, efficiency, furthering the self, enabling social goals, whether they are ultimately judged to be altruistic and beneficial or selfish and harmful.

Although Stinson (2020) distinguishes ontology, epistemology and ethics as dimensions of philosophy underpinning educational research paradigms, he argues that they cannot be neatly separated: "to speak about one always includes speaking implicitly or explicitly about the other two. … Philosophical engagement in general is complex and multilayered; it requires filigree ways of thinking with multiple and overlapping trajectories; ways of thinking that embrace uncertainty and openness rather than the fictions of certainty and closure." (Stinson 2020: pp. 11-12). He draws on Karen Barad's agential realism, which is at once an epistemology (theory of knowing), an ontology (theory of being), and an ethics. According to Barad, the deeply connected way that each 'thing' is entangled with everything else in materially specific ways means that all intra-actions reconfigure the entanglements. A researcher or agent performs a differentiating-entangling move that is enacted as an 'agential cut' that cuts things together-apart (one move) such that differences exist not as absolute separations but in their inseparability. Nothing is inherently separate from anything else, but separations are enacted within phenomena, our perceptions of materiality.

This view of knowledge provides a framework for thinking about how culture and habits of thought can make some things visible and other things easier to ignore or to never see. For this reason, according to Barad, agential realism is useful for feminist analysis and other forms of political and social thought, even if the connection to science is not apparent. In Barad's view, matter and meaning are co-constituted, inseparable, and becoming together. Just as matter and meaning cannot be separated, so too epistemology, ontology, and ethics cannot be thought apart. Researchers are "part of that nature that we wish to understand". (Barad, 2007, p. 26), we are becoming with when we research. Such thoughts have also been developed and extended to focus on the ethics of mathematics education, or rather a combined "ethico-onto-epistemology" of mathematics education (Paton and Sinclair 2024).[3]

Understanding the interconnectedness of all the questions, philosophies, research methodologies and interpersonal relationships and group thinking is part of an indigenous research paradigm developed at the Sitting Bull College. This is a tribal university chartered by Standing Rock Nation and guided by Dakota/Lakota culture, values, and language. The methodology is named Circulating Conversations Methodology and implements what came to be described as co-connecting knowledge (Luecke et al. 2022). In this paper the group of five authors describe both the process of developing the Circulating Conversations Methodology as an Indigenous Research Methodology and how this methodology was specifically enacted at Sitting Bull College to develop research questions for undergraduate mathematics education.

[3] Juan Godino et al (2024) offer an onto-semiotic approach that also includes and integrates epistemological and ethical dimensions for theorising mathematics education into an overall unified theory. However, this is done through the lens of cultural historical activity theory (CHAT) rather than through Barad's theoretical approach.

Part of the approach is that the researchers write in the first person and constitute a genuine 'we' together. Not surprisingly, the writing style is a central element of the methodology, for it aims to stay personal and connected, and is, in part, falsified by the present objectivised account. Honouring their tribal backgrounds and customs the researchers are bound together in a sharing open group. In this methodology the process to arrive at their research questions is equally as significant as the answer to the research questions. Similarly, the relationships developed through the process are equally as significant as the results.

Danny Luecke and his colleagues make it clear how different members join and participated in and influenced the research project as it unfolds (Luecke et al. 2022). To an outside commentator such as the present author, the central focus and glue that holds all the components together is clearly ethics. The honouring of tradition, of interpersonal relationships, the acknowledgement of the history of the nation, its colonial context, the democratic and cooperative modes of groups working all hinge on a deeply ethical outlook. Likewise, the goal of the researchers emerged as an ethical liberatory aim. To understand the relationships between

1. Western higher order mathematical concepts,
2. Lakota language and culture, and
3. Non-western higher order mathematics concepts.

The outcome is a clear articulation of an Indigenous research methodology, how it relates to the relevant literature, how it goes forward as a practice, and how the mathematics content of the college can be problematised and reconciled or contrasted with Western higher order mathematical concepts.

Like others, An Indigenous Research Paradigm has its own ontology, epistemology, methodology, and expected products. Building from that, the understanding of academic rigor is the *alignment* of all four of these categories. It is the agreement of ontology, epistemology, methodology, and products. However, research validity for an Indigenous Research Methodology/Paradigm does not come from statistical significance. It comes from the community's test of the researchers personally and the work they are doing. Will the community use it for their children at the pre-K to tribal college level? Meeting the Western standards of rigor but not showing respect to the relationships between researcher, participants, topic, Land, and community would be considered inauthentic or non-credible within Indigenous Research Methodologies.

Ethics

Ethics, as it is in the Indigenous Research Methodologies, has become a central topic in research in philosophy of mathematics education. For example, Ernest

(2018) challenges the idea that mathematics is an unqualified force for good. While acknowledging that mathematics is in many ways beneficial, he argues that learning mathematics can inadvertently cause harm, unless it is taught and applied carefully. Three ways in which mathematics causes collateral damage are described. First, the nature of pure of mathematics itself leads to instrumental and ethics-free thinking that can be socially damaging when applied beyond mathematics. Second the growing number applications of mathematics that underpin most areas of social functioning can be deleterious to our humanity, unless very carefully monitored and controlled. The abuses of democracy, and the risks of state and corporate control are well documented Third, the personal impact of learning mathematics on learners’ thinking and life chances can be negative for a minority of less successful students, including disproportionately many from minority backgrounds. Ernest recommends the inclusion of activities raising philosophical and ethical issues concerning mathematics alongside its teaching at all stages from school to university.

Dubbs (2020) attempts to Clarify the use of e*thics* in Mathematics Education Research, asking two related questions. First of all, how have theories of ethics been applied to mathematics education research? To answer this question, he reviews the philosophy of mathematics education literature, considering those articles which discuss ethics and mathematics education together. He finds the ethical perspectives adopted span normative and non-normative, and modern and postmodern orientations towards ethics.

The contrast of normative and non-normative enables Dubbs to contrast virtue, duty and utilitarian ethics with non-prescriptive ethics. Although these normative approaches to ethics have had little explicit uptake within mathematics education research, the questions these perspectives raise have been addressed unsystematically. Drawing on a practical ethics perspective on mathematics education research has pointed to anticipated beneficial consequences of the teaching and learning of mathematics, as well some negative consequences for a minority of students (e.g., Ernest 2018). These consequences have been used as a justification for aspects of mathematics education as well as, in a minority of cases, as a basis for critique.

The second question is what alternatives have not been considered, and what might the implications be if these alternative formulations were considered? He finds that ethics *per se* is construed too narrowly in the philosophy of mathematics education literature and considers that additional ethical perspectives from philosophy can be generative of new ideas.

The second distinction between modern and postmodern orientations towards ethics invites a critique of the majority of the literature reviewed, including

Ernest (2011). From a postmodern perspective there is a critical limitation within the mathematics education research. Philosophers of mathematics education continue to draw upon modern ethical formulations, which reduce ethical human experience to rule following. Notable exceptions include of Walshaw (2015) and Lawler (2012).

Instead, Dubbs draws on Butler’s (2015) ethical perspective. Butler argues that the ethical occurs concurrently with being, not prior to it. As was proposed earlier in the discussion of indigenous research methodologies, the argument is that the ethical, the ontological and the epistemological are inextricable. Butler claims that neither are we as humans reducible to the language with which we refer to our bodies nor are we bodies which transcend language. Instead, we are simultaneously constituted by language and constituting ourselves through bodily performances. Drawing on this perspective Dubbs poses new questions: How might we reimagine an equality between a teacher and their multiple students? How does mathematics education research respect the precarious humanity of those engaged in mathematics education (as students, teachers, researchers, etc.). How does language constrain and constitute the mathematical subject?

A recent collection (Ernest 2024a) encompasses these broader ideas and the wide spread of conceptualisations of ethics throughout the field of mathematics education and beyond, calling into question previously accepted boundaries, exclusions and connections from which mathematics has attempted to exclude itself. Indeed the “ethical turn” which has overtaken the humanities and philosophy has at last reached mathematics and mathematics education.

Critical Mathematics Education

The theme of critical mathematics education (CME) research is related to ethics – but it is primarily about social justice, a notion that is distinct from ethics. CME has several levels of focus. First there is the classroom. How can a critical pedagogy develop a critical approach to using and applying mathematics and foster the development of a critical citizenry. A numerate critical citizen should be able to read the mathematical presentations emanating from corporations, government, political actors and the media in order to evaluate their truth and distortions. This same citizen should also be empowered to use their knowledge to act in society to improve their own positions and opportunities, as well as to participate in group reforms. Clearly attempts to enact these ideas in practice, in and out of the classroom, are vital, but these belong to CME and not necessarily to its philosophy.

The second level is that of social critique and reform. CME offers a way of understanding society and the systems and algorithms in place that perform the distribution of wealth, knowledge, justice and social benefits. Coming to understanding this is the first step towards acting to change society for the better, little by little. Undeniably this is political. But the political unquestionably includes mathematics education as it is implicated as one of the means for the reproduction of society, just as mathematics, as the underlying conceptual framework, is implicated in the maldistribution of wealth.

The third level on which CME acts is in the critique of mathematics. There are many ways that mathematics can be presented that obstruct a critical view of the impact of mathematics on society. It can be seen as primarily pure, developed for its own sake, with no responsibility for its applications. It can be seen as a neutral tool, the uses of which are not the mathematician as technician's job to question. It can be seen as a universal knowledge created (or discovered) by Western civilisation providing the only way to reason logically and quantitatively. CME unpicks and offers deep critiques of these and other myths about mathematics. This is a philosophical critique, one that challenges some traditional philosophies of mathematics and epistemologies.

Much of CME is manifested as a practice, offering ways of organising the teaching and learning of mathematics and in more or less democratic forms with critical mathematical activities for learners to engage with. But its main interest here is as the philosophy of critical mathematics education. Many scholars have participated, including Ubi D'Ambrosio (1985), but the pre-eminent researcher in the philosophy of critical mathematics education is Ole Skosmose. He has been publishing a grand synthesis of his work in this area (Skovsmose 2023, 2024).

Skovsmose's (2023), 'A Philosophy of Critical Mathematics Education' offers a very wide view of the philosophical concerns of CME. In it he addresses the many concepts including *social justice*, *environmental justice*, *mathematics in action*, *foregrounds*, *dialogue,* and *critique*. Social justice and environmental justice embody visions with respect to the socio-political context within which learning takes place. In particular, how to make the human and material worlds better and more ethical places. Mathematics in action refers to possible social roles played by mathematics, and to how mathematics and power might be interrelated. Foreground, a term coined by Skovsmose, signifies features of the students' and teachers' life-conditions, including the width of their prospects and hopes. It is the mirror image of background, but future instead of past orientated. Dialogue and critique describe qualities of the process of learning, and how this might lead to political positionings.

Skovemose's goal is to enable the learning mathematics to change structural features of learner foregrounds and life-worlds. He offers explicit examples of where this has been done successfully. But changing social structures is a political act. Thus, the conceptual circle that he set out to explore has been completed. Learning mathematics is not just a classroom activity. It is a way of being, and a way of being political.

Skovsmose's work introduces new ideas which achieve their full expression in this work. He includes a philosophy of hope, a dimension of learner lifeworlds that is rarely referred to in the mathematics education literature. Hope is necessary because if one wants to put the learner centre stage in mathematics education, it is not enough to feed them reasons and justifications to motivate their study. They have to be authors of their own goals, motivations and directions of action, all of which require hope. For to act willingly they need the hope of a good future and a better world.

Introducing the idea of risk into mathematics education is another innovation. It is a topic especially susceptible to mathematical analysis and so is a valuable addition to the curriculum. But risk is powerful lens with which to see the world especially in these times of political, military and environmental crisis. School aged kids like Greta Thunberg and Malala Yousafzai are pointing out these risks and demanding that society takes them seriously, and that we ought to be addressing this in schools. We must learn from our students as we hope they learn from us. This is reflected in the dialogicality that Skovsmose proposes as a way to democratise education.

A central question posed is how can we locate mathematics in our life-worlds, in our immediate daily life experiences? Skovsmose's answering to this apparently straightforward question turns out to be quite complicated due to two interconnected processes: mathematisation and demathematisation. The mathematisation of the world means that much of our experience of the world, including mobile phone and computer Apps, shopping and travelling by public transport, and so on, depends on the complex mathematical algorithms and software that structures and performs such functions for us. The demathematisation is the process whereby this mathematisation is invisible and does not appear to be part of daily life practices. These two interconnected processes lead to a paradox. Modern life in virtually all of its organised aspects depends deeply on the widespread and near universal mathematisation of social functions based on a myriad of interconnected algorithms. However, the human interface of this underpinning software is increasing hidden, so that mathematics appears less frequently in our lifeworld and experience. An outcome is the belief that "mathematics is everywhere" is being replace by it appearing to be "nowhere". Most students do not see mathematics as something important in their lives, and for students in marginalised positions, mathematics only appears in an opaque format in their foregrounds. This is due to both to processes of demathematisation, but also because their daily life problems are so urgent that their foregrounds can be ruined.

How can mathematics disappear from in this way? Skovmose describes the banality of Mathematical Expertise, inspired by Arendt's use of the term, because of the ways applications are so often seen just as the outcomes of technical expertise, with no attention to ethics. But mathematics is deeply implicated in the quality of social life through its hidden formatting of social systems and mechanisms. Mathematics is performative in enacting all manner of fiscal and social policies that reshape modern societies (Ernest 2019). The Critical perspective on the philosophy of mathematics is greatly deepened by Skosmose's (2024) subsequent book (Critical Philosophy of Mathematics) which puts mathematics itself, never mind its powerful impact mediated by the theory and practices of mathematics education, under the microscope.

Thus Skovsmose (2023, 2024) provides a much-needed reconceptualisation of the role of mathematics in society. What is offered also is a philosophy of applied mathematics, something that the overwhelming attention to pure mathematics in philosophy has overlooked. Many more innovations in these works could be explored, but there is no room here to fully explore the philosophical (or pedagogical) implications of this philosophy of CME and CPM. All that is possible is to show that it is a vitally important and growing area of work within the Philosophy of Mathematics Education. It has the power to and is in the process of reshaping our ideas about the relationship between mathematics, society and education.

In offering a philosophy of applied mathematics within his overall frame, Skovsmose also address the problems of the ontology of mathematics. He offers his social constructivist perspectives on the questions: What is a mathematical entity? In what sense does mathematics exist? Where is mathematics? These questions signal another important growth area in the philosophy of mathematics education.

Ontology

In Ernest (2011) ontology was indicated out as a potential foundational base for mathematics education, but one which had not yet been much developed. Since then, growing attention has been directed towards ontology. Of course, in the philosophy of mathematics, Platonism and ontology have long been discussed (see, e.g., Bernays 1935, Cellucci 2020).

Within educational research, attention has been paid to ontology (as well as epistemology and ethics) as constituting the underpinning assumptions of a research paradigm. Ontology inquires into the kinds of objects we take for granted as populating the universe we are researching. Do we study material or measurable objects in physical space, as in scientific research; persons and their meanings, interactions and relationships, as in interpretative research; or institutions, power relationships and social change, as in critical paradigm research? Each of these three research paradigms presupposes a different sort of existent and existence on which to base their enquiry. An element of the research focus is the unit of analysis, the fundamental ontological molecule of study. It is a prototype or microcosm that represents the key relationships as well as the entities of a study (Ernest 2016). The unit "designates a product of analysis that *possesses all the basic characteristics of the whole*. The unit is a vital and irreducible part of the whole." Vygotsky (1987: 46, original emphasis). Various units of analysis of analysis have been employed by researchers. Vygotsky himself used *word meaning* as such a unit in his earlier work, and (socially embedded) *tool-mediated action* in his later work. Ernest (2016) uses *persons in conversation* as the unit in his account.

However, it should be noted that in addition to this ontological use or meaning of the term, the unit of analysis can be used methodologically, as a convenient object of analysis in a research project, without such strong ontological commitments.

Overall, the place of ontology in mathematics education is multiple. It concerns what there is – that is – being and becoming. It encompasses what kind of world is presumed to exist, the nature of social institutions within this world, how human beings are constituted, and what the objects of mathematics constituted are of (Ernest 2023).

Whatever theoretical stance is adopted the primary objects of study in mathematics education, whether or not the units of analysis, are human beings and their activities and aspects of their relationships. Ontology poses the question: what is a human being? Philosophically this is a very fundamental question. Answering it with respect to our field of study brings up issues of identity, subjectivity, agency, and human 'nature' and development. What is a *human* being? And what is human *being*? Currently, identity and its historical trajectory, in particular the learning career of students (and teachers), is a major area of research in our field.

Graven and Heyd-Metzuyanim (2019) indicate that there is ambiguity in the term 'identity'. They draw on the distinct meanings of identification and categorization; self-understanding and social location, and commonality, connectedness, groupness, and argue that the term identity subsumes multiple meanings. Their thorough review of research in this area of mathematics education shows how many distinct theoretical bases, conceptualisations, and operationalisations are in play. They conclude that many theorizations are based on the ideas of G.H. Mead, in which mind and self are fundamentally dialogical. Consequently, much of the research turns to narratives as fundamental to identity, which then facilitates research methodologies that are narrative based.

However, this local tradition, that is mathematics education research, primarily draws on social and psychological perspectives and methodologies. This perhaps suits the problems as perceived within mathematics education. But this work does not reflect an awareness of the broader philosophical work on identity.

In philosophy, the problem of personal identity is concerned with how one is able to identify a and characterise a single person. The *synchronic problem* concerns the question of what features and traits characterize a person at a given time. The diachronic problem of personal identity concerns the persistence of identity. In mathematics we take A=A for granted as an axiom of mathematical identity or equality. But if A is Anne, is Anne the same next year as she is today? It is clear that Anne today is not identical with Anne last year, but what persists and what changes over this period of time? This problem is very salient for mathematics education, for if we put Anne in mathematics class for a period, where we deliberately try to change Anne by engaging her in specially designed (mathematical) activities, what happens? Of course, there is a vast literature on pedagogy (causes of change) and assessment (measures of outcome changes). Indeed, measures of personal change are termed ipsitive assessment (Talbott and Horst 1960).

There are multiple philosophical traditions that address issues of personal identity and subjectivity. The phenomenological tradition is one, drawing on Kierkegaard, Husserl, Heidegger, Merleau-Ponty and others. One of the concepts developed in this tradition is that of ‘lifeworld’ the world of lived experience inhabited by us as conscious beings. Skovsmose (2023) draws on this perspective in order to take a learner-centred view of the experience and possibilities open to students of mathematics. He uses the terms background and foreground to describe a person’s views of their future and past as they can perceive them with the aid of their imagination, within the constraints and opportunities imposed by their social context and environment. Maria Bicudo has long employed and championed phenomenological approaches to mathematics education, including her recent reconfiguration of the nature and possibilities of the experience and production of mathematics in cyberspace (Bicudo 2021). One of the central features is how human intentionality is represented and expressed in cyberspace as an enlargement of our lifeworlds.

Heidegger (1962) develops a complex metaphysics of being based on the idea that our understanding of ourselves and our world presupposes something that cannot be fully articulated, a kind of knowing-how rather than a knowing-that. At the deepest level, such knowing is embodied in our social skills, in how we interact with and share experiences and practices with others, rather than in our concepts, beliefs, and values. Heidegger argues that it is these cultural practices that make our lives meaningful and give us our identities. In one formulation of his ontology Heidegger distinguishes three types of being: human, objects, and 'ready to hand' objects (Bagni 2010). These last are tools, shaped objects in our world embodying human intentions. Already this ontology suggests a novel way of conceptualizing technology as a special type of existents sculpted from inert matter but imbued with human intentionality.

Ernest (2024c) addresses what are termed the ontological problems of mathematics and mathematics education together. The ontological problem of mathematics is that of accounting for the nature of mathematical objects. The ontological problem of mathematics education concerns the chief entities in the domain, namely persons. What is the nature of persons, restricting the inquiry to the nature of their mathematical identities and their associated powers? Adopting conversation theory as the fundamental mechanism through which both mathematical objects and persons as mathematicians are socially constructed it is argued that the two ontological problems, converge. It is claimed that the rules and conventions of mathematical culture help build up and constitute both of these types of entity. The objects of mathematics are abstracted actions encapsulating such mathematical rules. Mathematical identities are constituted, shaped and constrained through the internalization and appropriation of these rules. In consequence, is claimed that the necessity that is so characteristic of mathematics is deontic. Mathematical tasks and mathematical texts primarily employ the imperative mode, and mathematical necessity is thus based on the rules, customs and norms of the institution of mathematics rather than some externally sourced necessity.

Ernest (2023, 2024b) argues that the formation of the mathematical identities of students of mathematics and mathematicians, which make up only a part of their overall beings as persons, develop through mathematical enculturation. The key element of this process is subjection to rules, conventions, orders, instructions that must be obeyed, at three levels, during engagement with mathematical activities.

First there is the social, interpersonal level. In schooling the teacher sets tasks and goals. However, they may be hedged, the teacher issues orders to the children that requires that they engage in the set mathematical activities or tasks. The teacher also demonstrates and reinforces the rules and solution processes that the learners must use to attempt to achieve these goals. There may be a limited degree of flexibility as in some tasks the learner can select their preferred method of solution from among the approved methods or their variations. But overall, this is the level made up of the imperatives issued directly by the teacher in social or . interpersonal space.

The second level of necessity is that inscribed within the texts of the tasks. The most common verb forms in mathematics, both in school and research texts, are imperatives requiring the reader to complete the activity in prescribed ways (Ernest 2018a, Rotman 1993). Such prescriptions may be tacit, but there is a repertoire of agreed rules and methods to be employed. Here the key characteristic is that the imperatives are in the text themselves.

Third, there are the tacit and explicit rules and conventions of mathematics that delimit the permitted actions and textual transformations. These are part of the culture of mathematics and a key element of what students and practitioners pick up and internalise as a residue of the myriad conversational exchanges in the dialogic space of mathematics. These make up much of what is termed the knowledge of mathematics, that which is learned through mathematics education. It is these rules that must be selected from and utilized in the performance of mathematical activities and tasks by students of mathematics and mathematicians.

What this account suggests is that mathematical identities, in the sense employed, are formed and shaped by power, expressed in interpersonal directives and compulsions, explicit, tacit and textual. How this can be tested empirically is another matter, but anthropology, social sciences and mathematics education are, between them, rich with research methods and methodologies. What it does demonstrate is the value of a philosophical perspective on the problems of mathematics education, adopting theories from metaphysics, ontology and the philosophy of mathematics.

Aesthetics

One of the more recent areas of research in philosophy of mathematics education concerns the role of aesthetics in mathematics and mathematics education. A prominent researcher in this area is Nathalie Sinclair who co-edited a volume on the affinity between aesthetics and mathematics (Sinclair and Higginson 2006) as well as developing aesthetic approaches to teaching children mathematics (Sinclair 2006). More recently she has signalled an aesthetic turn in mathematics education (Sinclair 2018). Ernest (2015) also investigates the nature and types of beauty in mathematics. In Ernest (Forth.) he argues that beauty, and more generally aesthetics, is more important in mathematics than is often credited. Yes, there is beauty in the creations of mathematicians and in visual representations of mathematics, just as in all of the arts and crafts, as well as in intellectual endeavours, which can all be appreciated for their beauty. But more importantly, in the creation of mathematics aesthetic considerations are what drive choices. Creative mathematics is both about necessity, following rules, but also about choice over which problems, method, concepts, axioms and theories (and hence rules) to investigate, and which results, proofs and theories are preferred. The choice criterion is aesthetic. As in any pure art form, it is mostly beauty but also interest and other aesthetic criteria that drive the creator. Mathematics depends on aesthetics in its making, research practices, and its appreciation. Controversially it is argued in Ernest (forth.) that given correctness and consistency, beauty and aesthetics play a larger role in mathematics than truth.

Conclusion

Many areas of work that could be drawn into this overview have been left out for space reasons. There is valuable philosophical work being done in mathematics education from the perspectives of Critical race theory, CHAT, Embodiment, Ethnomathematics, Feminist theory, Hermeneutics, Learning theory, Noticing, Phenomenology, Post colonial theory, Post humanism, Semiotics, Social Justice, Wisdom and doubtless further areas. A new body named The Critical Philosophical and Psychoanalytic Institute for Mathematics Education (CPPI-ME) has been set up, publishing its own Journal for Theoretical & Marginal Mathematics Education. This complements the long standing The Philosophy of Mathematics Education journal, which at the time of writing, is in its 35th year of publication. The Mathematics Education and Society (MES) group set up because of the exclusion of political and social problems as relevant research for the international Psychology of Mathematics Education (PME) research group, also welcomes and discusses philosophical research in mathematics education.

This paper has drawn together different selections of current research in the Philosophy of Mathematics Education. What is offered is not a map, it is a sampling from a rich and growing if loosely defined subspecialism. It cannot even be used to make the claim that philosophy should be used in any piece of mathematics education research, beyond understanding the standpoint of the research paradigm employed. For what background disciplines one draws upon must depend on the problems and issues one investigates. What it does show are two things. First that a growing body of research in our field does draw on philosophical theories. Second, that a philosophical perspective can be used fruitfully for some areas of inquiry, and that it throws up new ways on investigating research problems that might be overlooked if one stuck to the more familiar mathematical, psychological or social theoretic research backgrounds. It is not too early to claim that a ‘philosophical turn’ can be discerned in mathematics education research, as well as glimmerings of an ‘ethical turn’, as the publication of the first monograph on ethics and mathematics education indicates (Ernest 2024a).

Beyond the Philosophy of Mathematics Education journal, the quadrennial International Congress of Mathematical Education (ICME) has regular philosophy of mathematics education groups producing volumes of research reflecting the health, growth and breadth of the area, beginning with Ernest (1994a, b) and more recently represented by Ernest (2018) Bicudo et al. (2023), and Czarnocha, et al. (Forth.).

For those who fear that philosophy is the ultimate theoretical approach, and that as theory draws us away from the practices and practicalities of teaching and learning mathematics, we may recall the psychologist Kurt Lewin’s (1952) quote "There is nothing so practical as a good theory." (p. 169).

References

Bagni, G. T. (2010) Mathematics and Positive Sciences: A Reflection following Heidegger. *Educational Studies in Mathematics*, Vol. 73, No. 1 (Jan. 2010): pp. 75-85.

Barad, K. (2007). *Meeting the universe halfway: Quantum physics and the entanglement of matter and meaning.* Durham, NC: Duke University Press.

Bernays, P. (1935). On Platonism in Mathematics. L’enseignement mathematique, Vol. 34 (1935), pp. 52–69. Translation in P. Benacerraf and H. Putnam, Eds, *Philosophy of Mathematics: Selected readings*, Englewood Cliffs. New Jersey: Prentice-Hall, 1964: pp. 274-286.

Bicudo, M. A. V., Ed. (2021) *Constitution and Production of Mathematics in the Cyberspace A Phenomenological Approach*. Cham, Switzerland: Springer.

Bicudo. M. A. V. , Czarnocha, B., Rosa, M. and Marciniak, M., Eds. (2023). *Ongoing Advancements in Philosophy of Mathematics Education*. Cham, Switzerland: Springer

Burton, L. L. (2004). *Mathematicians as enquirers: Learning about learning mathematics* (Vol. 34). New York: Springer.

Butler, J. (2015) *Notes Toward a Performative Theory of Assembly*. Cambridge, MA, Harvard University Press.

Cellucci, C. (2020). The Nature of Mathematical Objects. B. Sriraman, (Ed.) *Handbook of the History and Philosophy of Mathematical Practice*. Cham, Switzerland: Springer.

Chronaki, A., Kollosche, D. (2019). Refusing mathematics: a discourse theory approach on the politics of identity work. *ZDM Mathematics Education* 51, 457–468 (2019). https://doi.org/10.1007/s11858-019-01028-w

Czarnocha, B., Degu, Y., Marciniak, M., Moller, R. and Thornton, S. Eds. (Forth.). *Exploring the Richness of Being Human through the Philosophy of Mathematics Education*, forthcoming.

D'Ambrosio, U. (1985). *Socio-cultural bases for Mathematics Education,* Campinas, Brazil: UNICAMP.

Davis, P. J. and Hersh, R. (1980). *The Mathematical Experience*. Boston: Birkhauser.

Dawson, A. J. (1969). *The Implications of the Work of Popper, Polya, and Lakatos for a Model of Mathematics Instruction,* unpublished Ph.D. thesis, University of Alberta.

Dawson, A. J. (1971). A fallibilistic model for instruction, *Journal of Structural Learning*, Vol. 3, No. 1: pp. 1-19.

DeFranco, T. C. (1996). A perspective on mathematical problem-solving expertise based on the performances of male Ph. D. mathematicians. *Research in Collegiate Mathematics Education, II,* 195–213.

Dewey, J. (1916). *Democracy and society*. New York: Macmillan.

Dubbs, C. (2020) Whose Ethics? Toward Clarifying *Ethics* in Mathematics Education Research. *Journal of the Philosophy of Education*, Vol. 54, No. 3 (June 2020): pp. 521-540

Ernest, P, Ed. (2018). *The Philosophy of Mathematics Education Today*, Switzerland: Springer international.

Ernest, P. (1991) *The Philosophy of Mathematics Education*, London, The Falmer Press.

Ernest, P. (2012) What is our First Philosophy in Mathematics Education?. *For the Learning of Mathematics.* Vol. 32 no. 3: pp. 8-14.

Ernest, P. (2014). Policy Debates in Mathematics Education. In: Lerman, S. (eds) Encyclopedia of Mathematics Education. Dordrecht: Springer. https://doi.org/10.1007/978-94-007-4978-8_125

Ernest, P. (2015). Mathematics and Beauty, *Mathematics Teaching*, No. 248 (September 2015), pp. 23-27.

Ernest, P. (2016) The Unit of Analysis in Mathematics Education: Bridging the Political-Technical Divide? *Educational Studies in Mathematics*.

Ernest, P. (2018) The Ethics of Mathematics: Is Mathematics Harmful?, in P. Ernest, Ed. (2018) *The Philosophy of Mathematics Education Today*, Switzerland: Springer international (pp. 187-216).

Ernest, P. (2019) Privilege, Power and Performativity: The Ethics of Mathematics in Society and Education. *Philosophy of Mathematics Education Journal*, No. 35 (December 2019)
Ernest, P. (2023). The Ontological Problems of Mathematics and Mathematics Education. Bicudo. M. A. V. et al, Eds. *Ongoing Advancements in Philosophy of Mathematics Education*. Cham, Switzerland: Springer.
Ernest, P. (2024b) The Ethics of Authority and Control in Mathematics Education: From Naked Power to Hidden Ideology. In P. Ernest, Ed. (2024a) *Ethics and Mathematics Education*, Switzerland: Springer international pp. 199-248.
Ernest, P. (Forth.). Is Beauty the First Value? The Case of Mathematics. In Czarnocha, B., Degu, Y., Marciniak, M., Moller, R. and Thornton, S. Eds. (Forth.). *Exploring the Richness of Being Human through the Philosophy of Mathematics Education*, forthcoming.
Ernest, P. Ed. (1994a) *Constructing Mathematical Knowledge: Epistemology and Mathematics Education*, London: Falmer Press
Ernest, P. Ed. (1994b) *Mathematics, Education and Philosophy: An International Perspective*, London: Falmer Press
Ernest, P. Ed. (2018) *The Philosophy of Mathematics Education Today,* Switzerland: Springer international.
Ernest, P. Ed. (2024a). *Ethics and Mathematics Education*. Cham, Switzerland: Springer.
Gödel, K. (1964) What is Cantor's Continuum Problem? in Benacerraf, P. and Putnam, H. Eds, *Philosophy of Mathematics: Selected Readings.* Englewood Cliffs, New Jersey: Prentice-Hall, 1964: 258-71.
Godino, J.D., Batanero, C., Burgos, M. et al. (2024). Understanding the onto-semiotic approach in mathematics education through the lens of the cultural historical activity theory. *ZDM Mathematics Education*, Vol. 56, pp. 1331–1344 (2024). https://doi.org/10.1007/s11858-024-01590-y
Graven, M., Heyd-Metzuyanim, E. (2019). Mathematics identity research: the state of the art and future directions. *ZDM Mathematics Education,* Vol. 51, No. 3 (June 2019): pp. 361–377. https://doi.org/10.1007/s11858-019-01050-y
Hersh, R. (1997). *What Is Mathematics, Really?* London, Jonathon Cape.
Heyd-Metzuyanim, E. (2019) Dialogue between discourses: Beliefs and identity in mathematics education. *For the Learning of Mathematics*, Vol. 39, No. 3 (November 2019): pp. 2-8
Kitcher, P. & Aspray, W. (1988) An Opinionated Introduction. In W. Aspray & P. Kitcher, (Eds.), *History and Philosophy of Modern Mathematics*, Minnesota Studies in the Philosophy of Science, Vol. XI). Minneapolis: University of Minnesota Press: pp. 3-57.

Lakatos, I. (1976). *Proofs and Refutations: The Logic of Mathematical Discovery* (Edited by J. Worrall and E. Zahar). Cambridge: Cambridge University Press.

Lawler, B. R. (2012) The Fabrication of Knowledge in Mathematics Education: A Postmodern Ethic Toward Social Justice, in: T. Cotton (ed.), *Towards an Education for Social Justice: Ethics Applied to Education*. New York: Peter Lang.

Lewin, K. (1952). *Field theory in social science: Selected theoretical papers by Kurt Lewin*. London: Tavistock.

Liljedahl, P. & Oesterle, S. (2020). Teacher Beliefs, Attitudes, and Self-Efficacy in Mathematics Education. Lerman, S. ed. *Encyclopedia of Mathematics Education* (2nd ed.). Cham, Switzerland: Springer. Pp. 825–828.

Luecke, D., Carlow, S., Mattes, J., Christensen, W. and Mackey, H. (2022). Circulating Conversations Methodology: Co-Connecting Knowledge to Develop Research Questions at Sitting Bull College. *Philosophy of Mathematics Education Journal*, No. 39 (September 2022). Accessed 21 November 2022 at https://education.exeter.ac.uk/research/centres/stem/publications/pmej/

Paton, K. and Sinclair N. (2024) An Ethico-Onto-Epistemology f or Mathematics Education. In P. Ernest (ed.), *Ethics and Mathematics Education*. Cham, Switzerland: Springer Nature. 2024, pp. 55-70.

Polya G (1957). *How to Solve It: A New Aspect* (2nd ed.). Princeton: Princeton University Press

Rocha H. (2019) Mathematical proof: from mathematics to school mathematics. *Philosophical Transactions of the. Royal society. Series A, Vol.* 377: pp. 1-12. http://dx.doi.org/10.1098/rsta.2018.0045

Schoenfeld, A. H. (1985). *Mathematical Problem Solving*. Orlando, Florida: Academic Press.

Sfard, A. & Prusak, A. (2005) Telling identities: in search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher* 34(4), 14–22.

Sinclair, N. (2006). *Mathematics and Beauty: Aesthetics approaches to teaching children*. New York: Teachers College Press.

Sinclair, N. (2018). An aesthetic turn in mathematics education. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 51-66). Umeå, Sweden: PME.

Sinclair, N. and Higginson, W. (2006) *Mathematics and the Aesthetic: New Approaches to an Ancient Affinity*. Cham, Switzerland: Springer Nature.

Sinclair, N. and Higginson, W. (2006) *Mathematics and the Aesthetic: New Approaches to an Ancient Affinity*. Switzerland: Springer international.

Skott, J. (2015) Towards a participatory approach to “beliefs” in mathematics education. In Pepin, B. & Roesken-Winter, B. (Eds.) *From Beliefs to*

Dynamic Affect Systems in Mathematics Education, Cham, Switzerland: Springer: pp. 3–24.

Skovsmose, O. (2023) *A Philosophy of Critical Mathematics Education*. Cham, Switzerland: Springer.'

Skovsmose, O. (2024) *Critical Philosophy of Mathematics*. Cham, Switzerland: Springer.'

Sriraman, B. (2020) *Handbook of the History and Philosophy of Mathematical Practice*. Cham, Switzerland: Springer.

Stillman, G., Brown, J. & Czocher, J. (2020). Yes, mathematicians do X so students should do X, but it's not the X you think!. *ZDM Mathematics Education, Vol.* 52, pp.:1211–1222 (2020). https://doi.org/10.1007/s11858-020-01183-5

Stinson, D. W. (2020). Philosophical considerations always already entangled in mathematics education research Mathematics *Teaching–Research Journal (Special issue on Philosophy of Mathematics Education)*, Vol. 12, No. 2: pp. 8–23. Retrieved 18 November 2022 from https://commons.hostos.cuny.edu/mtrj/wp-content/uploads/sites/30/2020/09/v12n2-Philosophicalconsiderations-always-already-entangled.pdf

Talbott, R. D. and Horst, P. (1960). *The multiple predictive efficiency of ipsative and normative personality measures*. Seattle: University of Washington.

Walshaw, M. (2013). Post-structuralism and Ethical Practical Action: Issues of Identity and Power. *Journal for Research in Mathematics Education*, 44.1, pp. 100–118.

Weber, K., Dawkins, P. & Mejía-Ramos, J.P. (2020). The relationship between mathematical practice and mathematics pedagogy in mathematics education research. *ZDM Mathematics Education* 52, 1063–1074 (2020). https://doi.org/10.1007/s11858-020-01173-7