DIDACTIC PRACTICES IN PRESERVICE TEACHERS' COLLABORATIVE WORK WITH PROBLEM SOLVING: A VARIATION THEORY PERSPECTIVE

Constanta Olteanu, Anna Lövström and Monica Jonsson

Linnaeus University, Sweden
constanta.olteanu@lnu.se

ABSTRACT

This study explores how preservice teachers (PSTs) interpret and enact variation theory (VT) in practice when collaboratively planning, enacting, and reflecting on problem-solving instruction in a mathematics education course. Twenty-two first-year PSTs participated in a course where VT served as a theoretical framework for conceptualising learning as the discernment of critical aspects and as a didactic resource guiding instructional design. The empirical material includes audio-recorded collaborative planning and reflection sessions, video-recorded enactments of problem-solving activities, and written and oral reflections. VT was used analytically to examine how theoretical principles were enacted in practice, focusing on how PSTs identify critical aspects, operationalise patterns of variation and invariance, and use representations throughout the instructional cycle. The findings show that PSTs' work is organised through three interrelated didactic practices, problem interpretation, representation, and justification, which shape how instructional purposes are articulated and translated into instructional design. Certain groups implemented VT systematically by aligning instructional purposes with coherent design. Other groups only partially implemented VT, resulting in discrepancies between their intended purposes and the enacted instruction. The collaboration and reflection during PSTs planning, enactment, and reflection supported negotiation, re-theorisation, and justification of instructional choices, and facilitated the translation of theoretical principles into practical application. This study contributes valuable insights to the body of research focused on theory-informed didactic practice, particularly in the realm of problem-solving instruction.

Keywords: Preservice teachers, problem-solving, didactic practices, variation theory, professional theorising, collaborative learning

Introduction

The concept of *didactic strategies* is well established in mathematics education. It is commonly used to describe teachers' deliberate choices concerning activities, representations, and forms of interaction aimed at supporting students' mathematical understanding (Jones & Tanner, 2002; Valcke et al., 2010). In this article, we retain this focus but use the term *didactic practices* to emphasise that such choices are not only planned in advance, but also enacted and reflected upon by PSTs in interaction with mathematical content, peers, and theory-informed instructional decisions. In particular, we distinguish between *instructional purposes*, referring to the overarching didactic intentions guiding PSTs' choices (why a particular problem, representation, or sequence is selected), and *instructional design*, which concerns the concrete organisation and sequencing of problems (how these purposes are operationalised into coherent instructional sequences). This distinction clarifies the difference

between didactic strategies and students' problem-solving strategies, which refer to their cognitive, metacognitive, and heuristic approaches to solving mathematical problems (Pólya, 1957; Schoenfeld, 1985). Didactic practices thus operate at the level of instructional purpose and design by shaping the conditions for learning. Problem-solving strategies characterise learners' activity within those conditions.

In the present study, *didactic practices* are investigated in relation to problem-solving and problem-posing, two closely connected domains that are strongly emphasised in both research and curriculum documents (The Swedish National Agency for Education, 2024; Schoenfeld, 2020). Mathematical problems are often described as non-routine and cognitively demanding (e.g., Schoenfeld, 1992). However, whether a task is experienced as a problem depends on learners' prior experiences and available solution strategies. For this reason, the problems addressed in this study are not defined a priori as non-routine but are analysed in relation to the PSTs' mathematical experiences and instructional purposes. Problem-posing involves generating new problems from given information or reformulating existing problems to make mathematical structures and relationships more explicit (Silber & Cai, 2021). However, integrating problem-solving and problem-posing presents well documented challenges for PSTs (Ball et al., 2008; Shulman, 1986). While previous studies have highlighted PSTs' difficulties with problem-solving and problem-posing (Avcu & Avcu, 2010; Cai et al., 2022; Silber & Cai, 2021), less attention has been paid to how PSTs operationalise theoretical principles in practice in relation to these types of problems.

Addressing this gap requires analytical approaches that reveal how theoretical principles shape instructional purposes and guide the enactment of theory-informed teaching over time. In this study, didactic practices are conceptualised as emergent and relational, that is, continuously shaped through iterative cycles of planning, enactment, and reflection. To explore how these principles are operationalised in practice, VT (Marton, 2015) is foregrounded as the philosophical and didactic lens, emphasising the discernment of critical aspects in learning and the deliberate use of variation and invariance to structure understanding. Combined with the notion of professional theorising (Ertsås & Irgens, 2016), this perspective enables analysis of how PSTs reason about and justify their instructional purposes and design. VT thus functions as a guiding perspective for setting instructional purposes and as an analytical lens for examining how PSTs translate theoretical principles into enacted instructional design.

The purpose of this study is to contribute to understanding how PSTs collaboratively develop and enact didactic practices informed by variation theory by examining how instructional purposes are articulated, operationalised, and negotiated through instructional design across planning, enactment, and reflection. The study is guided by the following research question: How do preservice teachers interpret and enact variation theory in practice when collaboratively planning, enacting, and reflecting on problem-solving instruction in a mathematics education course?

Related Literature

Problem Solving, Problem Posing, and Collaboration

Research consistently reports that PSTs face challenges in both problem-solving and problem-posing, particularly in designing rich problems that are aligned with the instructional purposes (i.e., the didactic intentions guiding why particular problem, representations, or sequences are chosen) (Avcu & Avcu, 2010; Cai et al., 2022; Işık & Kar, 2012; Mallart et al., 2018; Özgen

& Alkan, 2012). Research also indicates that problem-posing can support conceptual understanding when reformulated to clarify underlying mathematical structures (de Ron et al., 2022; Polat & Özkaya, 2023; Silber & Cai, 2021). Although problem-posing has been recognised as a fundamental practice in mathematics education (NCTM, 2000; Yuan & Sriraman, 2011), empirical studies show that PSTs' posed problems often remain at a surface or procedural level (Cai et al., 2022; Silber & Cai, 2021). Problems that focus on comprehension or translation tend to be more accessible, whereas posing mathematically rich problems involving editing, selection, or structural variation is considerably more challenging (Christou et al., 2005). These results show that problem-posing competence requires sustained, structured support that links instructional purposes to concrete instructional design choices during teacher education.

Collaboration during teacher education courses has been identified as a key condition for supporting PSTs' professional learning (Darling-Hammond & Oakes, 2019). Working collaboratively enables PSTs to articulate, negotiate, and justify instructional decisions (Darling-Hammond & Oakes, 2019; Mårtensson, 2023; Royea & Nicol, 2018; Zsoldos-Marchis, 2015) and provides PSTs with concrete models of practice that they encounter only theoretically (Nevin et al., 2009). For example, researchers identify that collaborative problem-posing for real or simulated learners can yield higher-quality problems and deeper professional insights (Crespo, 2015). Also, collaborative problem-posing has substantial potential to support shared sense-making, constructive feedback, and the co-construction of mathematical problems (English et al., 2005; Schindler & Bakker, 2020). Research on collaborative problem-posing further suggests that posing problems for an authentic or simulated audience can promote qualitative changes in PSTs' instructional reasoning (Crespo, 2015). Rather than adapting problems to make them easier or to avoid students' errors, PSTs' adaptations may become less leading and more open-ended, with greater emphasis on mathematical structure and reasoning (Crespo & Sinclair, 2008).

Variation Theory in Mathematics Education and Teacher Education

VT has been widely used in mathematics education research as a framework for analysing classroom practice (e.g., Kullberg et al., 2024; Marton, 2015; Olteanu, 2007). Some results of this research indicate that the use of systematic patterns of variation and invariance supports students' discernment of critical aspects of mathematical content (e.g., Kullberg et al., 2024; Marton, 2015). Within this body of research, VT has primarily been applied to analyse enacted teaching or to guide the instructional design (i.e., the organisation and sequencing of mathematical tasks).

In teacher education, VT has been used as a didactic perspective to support teachers' teaching of mathematical content and as an analytical framework to analyse classroom teaching (Kullberg et al., 2024; Olteanu, 2007). Variation theory (Marton, 2015) provides a philosophically grounded perspective on learning, viewing it as the process of discerning critical aspects through organised patterns of variation and invariance (the theory is presented in detail in the next section). By foregrounding discernment, VT provides a framework for analysing how shifts in what is perceived as mathematically significant are reflected in instructional purposes and instructional design decisions. Studies involving in-service teachers indicate that engaging with VT can strengthen teachers' attention to a specific mathematical content (e.g., Holmqvist & Selin, 2019).

Research examining PSTs' use of VT is still limited (Kullberg et al., 2024). However, study results show that PSTs often find it difficult to translate theoretical concepts into practice, and

that ideas connected to VT generated during lesson planning are not always enacted or reflected upon in teaching (de Haan et al., 2025; Royea & Nicol, 2018). These difficulties can be explained by the need for PSTs to align theoretical principles with concrete instructional design while maintaining clear instructional goals (Olteanu, 2018). From a theoretical standpoint, these difficulties can be seen as epistemic disruptions that destabilise established ways of handling a specific mathematical content (Olteanu, 2025). These findings indicate the need for research to examine how theoretical principles are interpreted, negotiated, and enacted in the planning, teaching, and reflecting processes.

Theory: Philosophical and Didactic Foundations

This study is grounded in VT and the concept of professional theorising, which together provide a philosophical and analytical framework for examining didactic practices in teacher education. Rather than functioning as a catalogue of instructional techniques, theory is positioned as a way of conceptualising learning, teaching, and professional reasoning, and as a set of analytical tools for interpreting empirical material and the instructional purposes and designs that emerge in practice.

VT originates from phenomenographic research, which conceptualises learning as a qualitative change in how a phenomenon is experienced or understood (Marton & Booth, 1997). Philosophically, learning is understood as a transformation in how a phenomenon is discerned, rather than as the accumulation of information. A central epistemological assumption of VT is that understanding requires the discernment of critical aspects of the object of learning. Critical aspects are dimensions of the object of learning that students need to notice to deepen their understanding (Marton, 2015). When certain aspects of a phenomenon vary while others remain invariant, the invariant aspect becomes discernible (Marton, 2015). Critical aspects are therefore not treated as fixed properties of mathematical content but as relational features constituted through instructional purposes (i.e., what is intended to be made discernible) and learners' engagement.

From a didactic standpoint, VT offers principles for organising teaching in ways that make critical aspects available for learning. These principles concern how tasks are sequenced, how contrasts are introduced, and how representations are coordinated. Teaching is therefore conceptualised as the intentional organisation of variation, where instructional design serves as the means through which instructional purposes are operationalised, rather than as the transmission of procedures or explanations. In this study, VT is also used as an analytical lens to examine how didactic practices create or constrain learning opportunities. The analysis focuses on how PSTs identify critical aspects of mathematical content, how they operationalise patterns of variation and invariance, and how representations are used to support discernment across planning, enactment, and reflection as part of instructional design processes.

The patterns of variation support the discernment of critical aspects in different ways (Olteanu, 2016). *Contrast* highlights a critical feature by using a non-example. For example, by sequencing two similar-looking arithmetic word problems that require different operations to make the underlying problem structure visible to students. *Separation* is achieved by systematically varying one aspect while keeping the others constant, such as changing numerical values while maintaining the context and structure across tasks invariant. *Generalisation* involves using a set of tasks or problems in which the context varies, but the underlying structure remains invariant. *Similarity* is enacted when lesson plans incorporate multiple representations, such as verbal descriptions, drawings, bar models, and symbolic

expressions, to convey the same meaning. *Fusion* is reflected when multiple critical aspects, for instance, the problem structure, representation, and justification, are combined within a single activity that appears simultaneously in a problem-solving situation. This analytical use of VT enables the examination of how PSTs interpret, adapt, and negotiate theoretical principles in practice, instead of evaluating instructional planning against theoretical ideals.

To complement VT's focus on content and instructional planning, the study also draws on the concept of *professional theorising* (Ertsås & Irgens, 2016). Professional theorising conceptualises teachers' reasoning as operating across three interconnected levels. At level T1, reasoning is embedded in concrete instructional actions and contextual considerations. At level T2, practice is interpreted, and negotiated through collaborative activities, such as lesson planning or reflection. At level T3, theoretical concepts are explicitly invoked to justify, refine, or reframe instructional purposes and instructional design decisions. Rather than viewing these levels as hierarchical stages of development, professional theorising emphasises the dynamic movement between theory and practice. PSTs may move back and forth between levels as they plan, enact, and reflect on instruction.

This intercalated theoretical perspective allows for an examination of coherence and tension between philosophical assumptions about learning, didactic intentions (instructional purposes), enacted instruction (instructional design), and reflective re-articulation of theory-informed practices within mathematics teacher education, without reducing theory to either abstract principles or technical methods. Accordingly, didactic practices are examined through PSTs' task design, articulation of critical aspects, and reasoning about variation across these instructional processes.

Method

Context and Participants

The study was conducted within a first-year mathematics education course in a Swedish university teacher education programme. Twenty-two first-year PSTs participated. The course focused on arithmetic, with particular emphasis on problem-solving and problem-posing, and adopted VT (Marton, 2015) as a guiding didactic perspective. PSTs worked collaboratively in self-selected groups of three to four participants throughout the course. Within these groups, PSTs planned, enacted, and reflected on instructional activities intended for enactment within the group setting, rather than in actual classroom contexts. This structure provided a controlled environment to examine how PSTs operationalise theoretical principles by linking instructional purposes with instructional design decisions in practice.

Didactic Framework in the Course

VT (e.g., Marton, 2015) served as the primary didactic perspective, offering a conceptual framework to guide PSTs' planning and structuring of problem-solving tasks. Critical aspects, patterns of variation and invariance, and the use of representations in solving problems, were introduced through lectures and course materials. PSTs were encouraged to foreground critical aspects of arithmetic problems, sequence problem-solving tasks, and generate opportunities to pose problems to support learners' problem-solving ability in relation to clearly articulated instructional purposes. VT guided both the instructional work and the study's analytical approach, providing a framework for identifying critical aspects, organising tasks, and coordinating representations. In their instructional work, PSTs used VT to identify critical aspects, to sequence tasks, and to coordinate representations. This analytical use

enabled interpretation of how PSTs applied theoretical principles across planning, enactment, and reflection.

Data Sources

The empirical material consisted of audio-recorded group-based lesson plans and reflections, video-recorded enactments of problem-solving activities, and written and oral reflections produced after enactment. The data captured PSTs' didactic practices during these instructional phases, and the relationship between their instructional purposes and instructional design.

Analytical Framework

This study used a qualitative, theory-informed approach, combining categories derived from theory with the empirical material (Gibson & Brown, 2009). In analysing empirical data, VT (e.g., Marton, 2015) was applied analytically to explore how PSTs use patterns of variation and invariance to distinguish critical aspects and make instructional decisions across these phases. By distinguishing between the didactic and analytical uses of VT, the study considered both how PSTs drew on theoretical ideas in practice and how these ideas shed light on emerging didactic practices at the levels of purpose and design.

The concept of professional theorising (Ertsås & Irgens, 2016) was also used to capture how PSTs articulated, justified, and reflected on their instructional choices. This concept operates across three interrelated levels:

- T1: Concrete instructional actions and contextual considerations during enactment (instructional design in action);
- T2: Collaborative articulation, negotiation, and interpretation of instructional practice during planning (linking instructional design to instructional purpose);
- T3: Explicit use of theoretical concepts to justify, refine, or reframe instructional choices (articulation of instructional purposes).

VT (Marton, 2015; Olteanu, 2016) and professional theorising (Ertsås & Irgens, 2016) enabled the examination of didactic practices. Professional theorising was used as an interpretive lens to characterise how PSTs' reasoning moved between theoretical articulation, collaborative planning, and instructional action across the instructional cycle.

Analytical Procedure

The analysis followed an iterative, thematic coding process (Gibson & Brown, 2009). First, all data were read and viewed holistically to gain familiarity with each group's instructional cycle, including planning, enactment, and reflection. In a second step, data were coded with a focus on didactic practices related to problem-solving, the use of representations, and the structuring of patterns of variation. Building on the distinction between instructional purpose and instructional design introduced earlier, these dimensions were operationalised as complementary analytical categories within the coding scheme. Data were coded to identify patterns in how PSTs articulated instructional intentions and how these intentions were enacted through concrete design decisions across planning, enactment, and reflection. In the third step, the identified codes in step two were systematically linked to VT (e.g., Marton, 2015) concepts to explore how PSTs' instructional purpose and instructional design highlighted or obscured problem-solving opportunities. Patterns in PSTs' reasoning were further interpreted in relation to levels of professional theorising (T1–T3) (Ertsås & Irgens, 2016) to characterise how theoretical ideas were articulated, enacted, and reflected upon across planning, enactment, and reflection. The results of this third step formed three themes, which are presented in the Results section. Data were analysed at the group level to maintain

consistency across PSTs collaborative work. Coding was iterative, with collaborative discussions among the research team to ensure reliability and depth.

Trustworthiness and Ethical Considerations

Trustworthiness was addressed through triangulation across data sources, iterative analysis, and collaborative discussion of coding decisions (Lincoln & Guba, 1985; Shenton, 2004). Ethical guidelines established by the Swedish Research Council (2024) were strictly followed. Participation was voluntary, informed consent was obtained from all participants, and all data were anonymised to protect confidentiality.

Results

The results are structured around three overarching themes that capture PSTs' didactic practices across these instructional phases. These themes illuminate how VT informed both the instructional purposes articulated by PSTs (what aspects of learning were intended to be foregrounded) and the instructional design through which these purposes were operationalised in concrete tasks, representations, and sequences of problems. Together, the themes show how VT functioned both as a practical resource for teaching and as an analytical lens for examining theory-informed didactic practices.

Theme 1: Articulating Didactic Intentions in Planning

This theme foregrounds how PSTs' engagement with VT shaped the identification of critical aspects and the use of patterns of variation in articulating instructional purposes, with differences in how explicitly these were operationalised in instructional design.

Some groups explicitly articulated critical aspects and linked them to problem-solving strategies and representations, demonstrating a clear instructional purpose informed by VT and a strong theoretical awareness. For example:

> Do they understand the meaning of these lines? What they symbolise? (Group A)
>
> Understanding of what, why, and how the rule of three is used. Also, when to use it—all four question words. (Group E)

Other groups engaged with these concepts more implicitly or struggled to connect VT concepts to the problem, resulting in instructional purposes that were weakly articulated or only partially connected to the mathematical content.

> […] the four basic operations, the block method, how fractions work and relate to each other. But it just must be them. (Group B)
>
> Be able to interpret what the task is about. Identify what we actually know. Apply the operations. Understand the strategy. (Group C)
>
> […] the idea with critical aspects is to be able to identify a common denominator—the lowest common one. (Group D)

One group (F) struggled to understand the concept of critical aspects, leading to no explicit or implicit links to the problem or the solving process.

Patterns of variation were addressed differently across groups, revealing differences in how instructional purposes were translated into instructional design choices during planning. Some groups explicitly connected variation to the problem and to the strategy for solving it, demonstrating the use of theoretically grounded, intentional instructional design to support learning.

> […] I'm thinking of a generalisation, considering that you've had a similar task before. You've had previous sessions that were similar. So that you turn the way of thinking into a habit. (Group C)

> […] one aspect that varies is that in the different tasks, some numbers are extended and some are shortened. One varies while others remain invariant. All of them involve the comparison method. That is what stays constant. (Group D)

> PST 1: A generalisation when using the rule of three is that there are always three known values.
> PST 2: And one unknown.
> PST 1: And actually, also that the same unit always appears underneath each other. (Group E)

In two groups (B and F), patterns of variation were more closely related to the process of solving the problem, suggesting that although an instructional purpose may have been present, it was not systematically realised through instructional design. This implicit treatment suggests that PSTs recognised the importance of variation but struggled to operationalise it for instructional purposes.

> PST 1: Similarity… that's when we show different ways of expressing something. We show it with different representations, like the block model.
> PST 2: That should be one… Student 1: Yes, because during the explanation it becomes similarity, since we both express it with numbers and then show it with materials. So, it should partly be similarity. (Group B)

> PST 1: […] the pattern of variation… that we show what the similarities are and what the differences are.
> PST 2: […] and that it means the same thing, whether it's geometric shapes or written form. (Group F).

One group (A) did not mention patterns of variation at all, indicating a complete absence of planning to discern the critical aspects through their use.

The differences observed across groups show that PSTs' engagement with critical aspects and patterns of variation varies from explicit, theory-informed planning to more implicit or even absent consideration. Groups that clearly articulated these critical aspects and patterns of variation demonstrated a stronger alignment between theoretical reasoning (T3) and problem-specific planning (T1).

Theme 2: Operationalising Didactic Practices in Enactment

This theme foregrounds how PSTs translated articulated instructional purposes into practice during enactment, revealing differences in how systematically patterns of variation and critical aspects were realised through instructional design.

Systematic implementation was evident when invariant problem-solving strategies were maintained while varying numerical relationships or representations to foreground critical aspects, indicating coherence between instructional purpose and instructional design. Group D sequenced problems to preserve invariant comparison structures while varying numerical relationships, thereby directing attention to proportional reasoning.

> PST 1: Is it something we can see right away?
> PST 2: We can see that two pens cost 14 SEK.
> PST 1: Yes, we can apply that up there. The teacher circles the pens. (Group D)

Group E enacted the problem-solving strategy by posing a sequence of increasingly complex problems, thereby supporting the understanding of unit relationships and the structure of proportionality.

> PST 1: Couldn't you just switch places then? Teacher: What do you mean by that?

> PST 2: Write it as it's written in the text? Teacher: A very important rule. It doesn't matter what you start with. In the second part, it's important that the same unit is placed underneath each other. (Group E)

In both cases, systematic implementation reflects the use of variation patterns and careful problem sequencing, supported by problem-posing to foster learners' conceptual understanding. These practices indicate PSTs' growing capacity to translate theoretical knowledge into a coherent instructional purpose and to realise this purpose through instructional design. This transaction ensures that each sequence contributes to the systematic discernment of critical aspects.

Partial or incidental enactment revealed challenges in translating VT from planning into practice. This enactment is characterised by partial alignment with intended instructional purposes, representations introduced without clear links to the problem structure, or inconsistent use of patterns of variation. For example, in group C, the students articulate explicitly in their lesson plans that critical aspects should guide instruction, yet this was not evident during enactment: "Yes, we need to start from the critical aspects, I think, so that we try to address them." The remaining three groups did not use the previously identified critical aspects, indicating a limited translation of theoretical reasoning into practice.

Regarding patterns of variation, only two groups systematically enacted them. Group D structured a sequence of five problem-posing tasks, progressing from simple to complex. Systematic variation was applied by keeping the comparison method constant while gradually changing the quantities, thereby requiring students to extend their reasoning. Two examples of these problem-posing tasks are:

> Problem 1:
> 1 ice cream and 2 juices: 40 SEK
> 2 ice creams and 2 juices: 50 SEK
> How much does one ice cream cost? How much does one juice cost?
> No extension required
>
> Problem 2:
> 1 soda and 3 chocolate balls: 56 SEK
> 2 sodas and 4 chocolate balls: 88 SEK
> How much does one soda cost? How much does one chocolate ball cost?
> Extension required (Group D)

Group E applied systematic variation in problem-posing associated with problem-solving, progressing from a simple pancake task to a more complex candy-making task involving multiple ingredients and unit conversions. This progression is characterised by maintaining invariant the solving strategy (*regula de tri*) across problems. The PST scaffolded students' reasoning through guiding problem-posing, thereby supporting the solving strategy with both more straightforward and more complex problems.

> Problem 1:
> You can make 10 pancakes using 2 deciliters of flour. How many pancakes can be made using 6 deciliters of flour?
>
> Problem 2:
> Candy recipe (makes 60 pieces):
> 450 g granulated sugar
> 125 g glucose
> 1 dl water

> 2 tsp citric acid
> 2 ml food colouring
> 1.5 ml pear flavouring
>
> a) Kevin and Maja are making candies for the class to sell. One package of glucose contains 1 kg.
> How many candies can they make with one package?
> b) They have 360 g of granulated sugar left and want to use it to make candies.
> How much glucose do they need? (Group E)

These examples show how systematic variation operationalises theoretical intentions. At the same time, partial or incidental implementation highlights the gap between theory and practice, where PSTs may consider critical aspects and the use of patterns of variation in planning but fail to enact them consistently.

Theme 3: Reflecting on Didactic Practices

This theme foregrounds variation in PSTs' reflective engagement with VT following enactment, revealing differences in how explicitly critical aspects and patterns of variation were used to evaluate, justify, and refine instructional purposes and design.

Several groups engaged in reflective reasoning, connecting previously identified critical aspects to students' learning and the instructional context. This reasoning demonstrates metacognitive engagement and an awareness of the pedagogical implications of their planning. Group A considered potential student difficulties:

> PST 1: [...] it can be difficult for children to grasp what these lines actually mean.
> PST 2: [...] to understand how to use the block model and the bar method... that it's important the parts are equal in size and that you sort of build the parts together. (Group A)

Group B reflected on the importance of the block method as a means of structuring the problem-solving process, noting that it functioned as a critical aspect for understanding the solution: "Block method… became a kind of critical aspect; it's hard to follow the examples if you don't understand why we are drawing three bars here" (Group B). Similarly, Groups C and D emphasised task sequencing and proportional reasoning as central for structuring the solving process.

> The critical aspects I observed during our lesson were embedded within the block model itself. It is important to understand which block to base the reasoning on—for example, why the screws are represented by three blocks and not six. (Group C)
>
> […] to identify a relationship between the individuals and the number of items purchased, in order to understand that the task is solved by comparing goods and prices. (Group D)

Group E emphasised unit conversion as a critical aspect, noting that "the ability to convert units also emerged as a critical aspect—for example, converting from kilograms to grams" (Group E). In contrast, one group (F) struggled to grasp the concept of critical aspects, resulting in neither explicit nor implicit links to the task or the intended problem-solving strategy.

These examples show that reflections explicitly connecting critical aspects to teaching actions indicate greater theoretical coherence, as PSTs critically examined the alignment between instructional purpose, instructional design, and enacted practice. Groups practicing this type of reflection seemed better at linking theory (T3) and instructional action (T2). There are also

groups engaged in more superficial reflection, characterised by a lack of connecting them to critical aspects or to the underlying instructional purpose guiding the instructional design. For instance, Group A noted critical aspects but did not relate them to variation patterns or planning intentions.

Reflections also varied in terms of engagement with patterns of variation. In three groups (C, D, E), PSTs explicitly reasoned about patterns of variation in ways that connected directly to the problem-solving and strategy:

> [...] show counterexamples to clarify what the task is about. For example, that it's important to start from the nuts in the task. [...] you can start from the bolts to show students that it's not the easiest way. (Group C)
>
> [...] take one method at a time. Once they know this, you could move on with the same type of tasks but solve them in a different way. That also gives you variation. (Group D)
>
> In teaching *regula de tri*, the specific units used are irrelevant—as long as three quantities are known and one is unknown, the method will always work. I believe this generalisation is important for students to understand from the very beginning. (Group E)

In the remaining three groups (A, B, F), reflections on patterns of variation were more descriptive or fragmented, and the connection to the intended learning outcomes or overall strategy was less clear:

> PST 1: Which patterns of variation are we using, and why these ones?
> PST 2: We thought these lines would best help make the task more concrete, but really, it's better to use more methods... patterns.
> PST 1: Because the more you can make it concrete, the better. ...I mean, we did the calculations and then we drew the bars. (Group A)

> PST 1: ...we're including contrast.
> PST 2: Yeah, but are we really including that much contrast?
> PST 1: What fractions are and aren't? No, not in the actual video.
> PST 3: I don't think it's clear what a fraction is and isn't.
> PST 2: Then we would've needed to say something like "this is one third, and here we have three wholes"—what's the difference? (Group B)

> Contrast is a pattern of variation we could have used by drawing different geometric shapes and showing that, for example, one fourth can look different depending on the shape. (Group F)

Theme 3 illustrates that groups that linked their reflection (explicitly or implicitly) to the use of critical aspects and patterns of variation (T3) in their initial planning and enacted practices (T2) demonstrated an integrated understanding of how theoretical concepts from VT can guide instructional action (T1). In these cases, reflection functioned as a process through which PSTs evaluated the alignment between intended learning objectives, enacted instruction, and the use of problem-posing to structure problem-solving strategies and to discern critical aspects. In other cases, reflections that were not grounded in VT demonstrated limited capacity to critically evaluate and refine PSTs' didactic practices. These results indicate challenges in moving between T2 and T3, where experiences from enactment are expected to inform and deepen theoretical reasoning. The results also highlight the need to support PSTs in using reflection to strengthen coherence between theory, planning, and enactment within a variation-theoretical framework.

Synthesis of Didactic Practices Across Planning, Enactment, and Reflection

Taken together, the results answer the research question by showing how PSTs interpret and enact variation theory when collaboratively planning, implementing, and reflecting on problem-solving instruction in a mathematics education course. Across these instructional phases, PSTs' work is organised around three interrelated didactic practices (problem interpretation, representation, and justification) that capture how instructional purposes are articulated, operationalised, and negotiated over time. Rather than appearing as isolated actions, these practices are enacted throughout the instructional cycle, revealing how PSTs move between T3, T2, and T1.

Practices related to problem interpretation involve how PSTs collaboratively negotiate the meaning of a problem, identify what is given and what is sought, and delimit the mathematical content to be foregrounded. These practices are central to articulating instructional purposes, particularly by identifying critical aspects during planning. They also re-emerge during enactment and reflection, when PSTs reassess whether the intended focus supports learners' understanding and problem-solving strategies.

Practices related to representation concern how PSTs select, coordinate, and translate between symbolic, visual, and verbal representations as part of instructional design. Representations function as didactic resources for operationalising instructional purposes by making mathematical relationships visible, supporting solution processes, and scaffolding reasoning. Through shifts between representations, PSTs test the coherence between intended instructional purposes and enacted instructional design, particularly in relation to identified critical aspects.

Practices related to justification concern how PSTs explain and critically evaluate problem-solving processes, including why particular strategies are used and how instructional choices support students' discernment of critical aspects. In this way, justification operates as a collective practice that mediates the relation between VT (T3) and instructional design/enactment (T2–T1), ensuring that instructional purposes are coherently realised in enacted instruction.

These three didactic practices provide an account of how instructional purposes are translated into instructional design and enacted in collaborative problem-solving contexts. The synthesis shows how PSTs draw on VT and how its principles are interpreted, operationalised, and stabilised (or constrained) across planning, enactment, and reflection.

Discussion

This study contributes to understanding how PSTs develop and enact didactic practices informed by variation theory, particularly in relation to how instructional purposes are articulated and operationalised through instructional design. The results reveal that PSTs vary in the extent to which they explicitly articulate critical aspects. Some groups identified these aspects and connected them to problem-posing by keeping problem-solving strategies and representations invariant. These groups show strong alignment between theoretical reasoning (T3) and instructional purposes, realised through coherent instructional design choices. Other groups used critical aspects more implicitly without linking them to instructional actions. Such implicit use limited coherence across planning, enactment, and reflection and constrained the translation of VT into instructional design.

While VT (e.g., Marton, 2015) provided a productive analytical lens for examining PSTs' didactic practices, the findings also highlight its role as a framework for articulating instructional purposes and as a resource for instructional design. In several groups, critical aspects were treated as isolated rather than relational, that is, disconnected from systematic use of patterns of variation. This weakened PSTs' capacity to coordinate problem interpretation and representation as coherent didactic practices. This finding addresses the gap identified in the introduction by demonstrating how PSTs operationalise theory through instructional design, rather than merely referring to theoretical concepts in their planning. When critical aspects were coordinated with patterns of variation and representations of problem content were applied systematically, VT supported coherent problem sequencing. In contrast, when theoretical understanding was insufficient, VT application was partial or inconsistent, hindering the development of a coherent instructional design. Supporting PSTs in understanding the dynamic relationship between invariance and variance is therefore crucial for avoiding superficial applications of VT. To enable PSTs to apply VT consistently and ensure coherence between instructional design and intended instructional purposes, sustained, practice-based engagement with VT is necessary.

The transition from articulated planning guided by the VT principle (T3) to instructional action (T2) was fragile. Robust transitions were evident when PSTs systematically used patterns of variation in instructional design to realise instructional purposes (Groups D and E). Partial or incidental enactment weakened the alignment between instructional purposes and instructional design, revealing the fragility of transitions from theoretical reasoning (T3) to instructional action (T2). These findings build on prior research on PSTs' challenges in enacting problem-solving instruction (Polat & Özkaya, 2023; Silber & Cai, 2021), showing that such challenges are closely tied to the management of critical aspects, patterns of variation, in the didactic practice.

Reflection emerged as a critical phase for consolidating didactic practices. PSTs who critically examined enacted instruction in relation to critical aspects and patterns of variation were able to evaluate the coherence between instructional purposes and instructional design, strengthen theoretical reasoning, refine instructional decisions, and integrate learners' perspectives. Groups whose reflections remained descriptive or weakly connected to theory offered limited opportunities for professional learning. These findings indicate that justification functions as a central didactic practice, linking theory and instructional action by mediating between articulated purposes and enacted design. This result supports prior research emphasising the need for analytical tools that move beyond retrospective description (Christou et al., 2005; Crespo & Sinclair, 2008; Silber & Cai, 2021).

Collaboration served as a cognitive and social resource for PSTs. Through group discussions, PSTs externalised their reasoning, negotiated interpretations of VT, and jointly articulated instructional purposes and instructional design decisions. Collaborative interaction enabled re-theorisation, as theoretical ideas were adapted, transformed, and contextualised in response to instructional constraints. This highlights the epistemic work involved in operationalising theory by translating instructional purposes into feasible instructional design and adds nuance to research on collaborative teacher learning (Darling-Hammond & Oakes, 2019; Nevin et al., 2009; Zsoldos-Marchis, 2015). Taken together, reflection and collaboration functioned as mechanisms for integrating instructional purposes with instructional design and for sustaining coherence across planning, enactment, and reflection. PSTs' challenges in enacting problem-solving instruction were thus linked to how critical aspects and patterns of variation were coordinated within instructional design.

The findings foreground VT as both a philosophical framework for understanding learning and a mediating resource for articulating instructional purposes and guiding instructional design. Examining VT in practice reveals how its core principles, discernment of critical aspects through patterns of variation and invariance, are interpreted, adapted, and sometimes only partially operationalised by PSTs. These findings contribute to a nuanced understanding of theory–practice relations in mathematics teacher education, illustrating not only whether, but how, PSTs operationalise VT in collaborative problem-solving contexts.

Conclusion

This study provides insight into how PSTs operationalise VT in practice as they develop and enact didactic practices with arithmetic word problems. The findings indicate that variation theory (e.g., Marton, 2015) serves as a conceptual and practical framework, guiding the discernment of critical aspects through the use of patterns of variation. The use of patterns of variation are connected to the representation choices as part of coherent instructional design aligned with explicit instructional purposes. Collaborative planning, enactment, and reflection emerged as central for linking theoretical reasoning with instructional action. By examining VT in practice, the study shows how philosophical principles of discernment and variation are translated into concrete didactic actions.

Implications

The findings suggest that developing theory-informed didactic practices requires systematic support for PSTs across planning, enactment, and reflection, enabling them to articulate instructional purposes and operationalise these through instructional design that supports students' discernment of problem-solving processes and conceptual connections. Accordingly, teacher education programmes should provide opportunities for PSTs to identify and articulate critical aspects and to work explicitly with patterns of variation to ensure coherence in instructional practice and effective use of representations. In addition, programmes should foster analytically grounded reflection by creating collaborative spaces where theoretical ideas can be negotiated, re-theorised, and connected to practical instructional constraints.

Limitations

Several limitations should be acknowledged. First, the study was conducted in a simulated teaching context rather than with actual students, which may limit generalisability to classroom practice. Second, the sample comprised 22 first-year PSTs from a single teacher education programme, which may limit the transferability of the findings to other contexts or educational systems. Third, the study focused primarily on arithmetic word problems, so further research is needed to explore how PSTs develop didactic practices in other mathematical domains. Finally, while multiple data sources were used, the analysis relied on researchers' interpretation of discussions and enactments, which may introduce subjectivity despite measures to ensure trustworthiness.

Acknowledgements

The authors would like to thank the preservice teachers who participated in this study for their engagement and thoughtful contributions. This research was supported by the Linnaeus Knowledge Environment: Education in change, which enabled the study.

References

Avcu, S., & Avcu, R. (2010). Preservice elementary mathematics teachers' use of strategies in mathematical problem solving. *Procedia Social and Behavioral Sciences*, 9, 1282–1286. https://doi.org/10.1016/j.sbspro.2010.12.321

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching. What makes it special? *Journal of Teacher Education, 59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Cai, J., Koichu, B., Rott, B., Zazkis, R., Jiang, C. (2022). Mathematical problem posing: Task Variables, processes and products. In C. Fernandez, S. Linares, A. Gutierrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 119–145). PME.

Christou, C., Mousoulides, N., Pittalis, M., Pitta-Pantazi, D., & Sriraman, B. (2005). An empirical taxonomy of problem posing processes. *Zentralblatt für Didaktik der Mathematik* 37(3), 149–158. https://doi.org/10.1007/s11858-005-0004-6

Crespo, S. (2015). A collection of problem-posing experiences for prospective mathematics teachers that make a difference. In F. M. Singer, & N. F. Ellerton (Eds.), *Mathematical problem posing* (pp. 493–511). Springer.

Crespo, S., & Sinclair, N. (2008). What makes a problem mathematically interesting? Inviting prospective teachers to pose better problems. *Journal of Mathematics Teacher Education,11*(5), 395–415. https://doi.org/10.1023/A:1024364304664

Darling-Hammond, L., & Oakes, J. (2019). *Preparing teachers for deeper learning*. Harvard Education Press.

de Haan, D., de Vries, S., Roorda, G., & Drijvers, P. (2025). Applying variation theory: a challenging case in pre-service teacher education. *Proceedings of the Fourteenth Congress of the European Society for Research in Mathematics Education (CERME14)*, ERME; ERME, Feb 2025, Bozen-Bolzano, Italy. hal-05302455

de Ron, A., Christiansen, I., & Skog, K. (2022). Configuring the landscape of research on problem-solving in mathematics teacher education. *International Electronic Journal of Mathematics Education, 17*(4), em0712. https://doi.org/10.29333/iejme/12457

English, L. D., Fox, J. L., & Watters, J. J. (2005). Problem posing and solving with mathematical modeling. *Teaching Children Mathematics*, 12(3), 156–163, https://doi.org/10.5951/TCM.12.3.0156

Ertsås, T. I., & Irgens, E. J. (2016). Professional theorizing. *Teachers and teaching. Theory and practice*, 23(3), 332–351. https://doi.org/10.1080/13540602.2016.1205013

Gibson, W. J., & Brown, A. (2009). *Working with qualitative data*. Sage.

Holmqvist, M., Selin, P. (2019). What makes the difference? An empirical comparison of critical aspects identified in phenomenographic and variation theory analyses. *Palgrave Commun*. 5, 71. https://doi.org/10.1057/s41599-019-0284-z

Kullberg, A. & Ingerman, Å. & Marton, F. (2024). *Planning and Analyzing Teaching: Using the Variation Theory of Learning*. 10.4324/9781003194903

Işık, C., & Kar, T. (2012). An error analysis in division problems in fractions posed by preservice elementary mathematics teachers. *Educational Sciences: Theory & Practice*, 12(3), 2289-2309. www.edam.com.tr/estp

Jones, S., & Tanner, H. (2002). Teachers' interpretations of effective whole-class interactive teaching in secondary mathematics classrooms. *Educational Studies*, 28(3), 265–274. http://doi.org/10.1080/0305569022000003717

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Sage.

Mallart, A., Font, V., & Diez-Palomar, J. (2018). Case study on mathematics pre-service teachers' difficulties in problem posing. *Eurasia Journal of Mathematics, Science and Technology Education*. 14(4), 1465–1481. https://doi.org/10.29333/ejmste/83682

Marton, F. (2015). *Necessary conditions of learning*. Routledge.
Marton, F., & Booth, S. (1997). *Learning and awareness*. Lawrence Erlbaum.
Mårtensson, P. (2023). Professional theorizing: A key to understanding how prospective teachers improve their knowledge for teaching mathematics. *Proceedings of Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13)*, Alfréd Rényi Institute of Mathematics; Eötvös Loránd University of Budapest, Jul 2023, Budapest, Hungary. hal-04414758
National Council of Teachers of Mathematics. NCTM. (2000). *Curriculum and evaluation Standards for School Mathematics*., Reston, VA: National Council of Teachers of Mathematics.
Nevin, A.I., Thousand, J.S. & Villa, R.A. (2009). Collaborative teaching for teacher educators – what does the research say? *Teaching and Teacher Education*, 25, 569–574, https://doi.org/10.1016/j.tate.2009.02.009
Olteanu, C. (2025). Disorienting Dilemmas and Mathematical Reasoning: Exploring Transformative Learning in Preservice Teacher Education. *Journal of Transformative Learning,* 12(1), 33–49. https://jotl.uco.edu/index.php/jotl/article/view/535/402
Olteanu, C. (2018). Learning study – promoting and hindering factors in mathematics teaching. *International Journal of Mathematical Education in Science and Technology*, 49(7), 971–985. https://doi.org/10.1080/0020739X.2018.1443222
Olteanu, L. (2016). *Framgångsrik kommunikation i matematikklassrummet*. [Successful communication in the mathematics classroom]. Doctoral Thesis. Linnaeus University Press.
Olteanu, C. (2007). *Vad skulle x kunna vara? - Andragradsekvation och andragradsfunktion som objekt för lärande*. [What could x be? – Quadratic equations and quadratic functions as objects of learning]. Doctoral Thesis. Umeå: Umeå universitet.
Polat, H. & Özkaya, M. (2023). The effect of problem posing-based active learning activities on problem-solving and posing performance: The case of fractions. *Journal of Pedagogical Research*, 7(1), 67–81. https://doi.org/10.33902/JPR.202317880
Pólya, G. (1957). *How to solve it. A new aspect of mathematical method*. Doubleday and Company, Inc.
Royea, D. A., & Nicol, C. (2018). Pre-service teachers' experiences of learning study: learning with and using variation theory. *Educational Action Research*, 27(4), 564–580. https://doi-org.proxy.lnu.se/10.1080/09650792.2018.1515094
Schindler, M., & Bakker, A. (2020). Affective field during collaborative problem posing and problem solving: A case study. *Educational Studies in Mathematics*, 105, 303–324. https://doi.org/10.1007/s10649-020-09973-0
Schoenfeld, A. H. (2020). Reframing teacher knowledge: A research and development agenda. *ZDM–Mathematics Education*, 52(2), 359–376. https://doi.org/10.1007/s11858-019-01057-5
Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sensemaking in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334-370). Macmillan.
Schoenfeld, A. (1985). *Mathematical problem solving.* Academic Press.
Shenton, A. K. (2004). Strategies for ensuring trustworthiness in qualitative research projects. *Education for Information*, 22(2), 63–75. https://doi.org/10.3233/EFI-2004-22201
Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14. http://www.jstor.org/stable/1175860
Silber, S., & Cai, J. (2021). Exploring underprepared undergraduate students' mathematical problem posing. *ZDM-Mathematics Education, 53*, 877–89. https://doi.org/10.1007/s11858-021-01272-z

Swedish Research Council. (2024). *God forskningssed*. [Good Research Practice] https://www.vr.se/analys/rapporter/vara-rapporter/2024-10-02-god-forskningssed-2024.html

The Swedish National Agency for Education [Skolverket]. (2024). *Curriculum for Compulsory School, Preschool Class and School-Age Educare – Lgr22*. https://www.skolverket.se/sok-publikationer/publikationsserier/styrdokument/2024/curriculum-for-compulsory-school-preschool-class-and-school-age-educare---lgr22

Valcke, M., Sang, G., Rots, I., & Hermans, R. (2010). Taking prospective teachers' beliefs into account in teacher education. In P. Peterson, E. Baker, & B. McGaw (Eds.), *International encyclopedia of education* (7th ed., pp. 622–628). Oxford: Elsevier.

Yuan, X., Sriraman, B. (2011). An Exploratory Study of Relationships between Students' Creativity and Mathematical Problem-Posing Abilities. In: Sriraman, B., Lee, K.H. (Eds.) *The Elements of Creativity and Giftedness in Mathematics. Advances in Creativity and Giftedness*, vol. 1. SensePublishers. https://doi.org/10.1007/978-94-6091-439-3_2

Zsoldos-Marchis, I. (2015). Changing pre-service primary-school teachers' attitude towards mathematics by collaborative problem solving. *Procedia - Social and Behavioral Sciences*, 186, 174–182. https://doi.org/10.1016/j.sbspro.2015.04.100

Özgen, K., & Alkan, H. (2012). The relationship between secondary school pre-service mathematics teachers' skills in problem solving dimensions and their learning style characteristics. *Educational Sciences: Theory & Practice, 12*(2), 1173–1181. https://eric.ed.gov/?id=EJ981835