ON THE POSSIBILITY OF MATHEMATICAL ABSTRACTION

Felix Lensing and Thorsten Scheiner

Free University Berlin, Germany
lensing@zedat.fu-berlin.de
thorsten.scheiner@fu-berlin.de

ABSTRACT

In this paper, we develop an account of how mathematical abstraction is possible, as an essential part of an eidetic epistemology of mathematical knowledge. We first argue that acts of mathematical abstraction are central to the pursuit of mathematical knowledge, as they often 'give' or 'present' the objects with which mathematical propositions deal. Drawing on Husserl's distinction between ideal and real conditions of possibility of knowledge, we examine the ideal conditions of possibility underlying mathematical abstraction. We distinguish two essential types of acts—generalizing abstraction and formalizing abstraction—and analyze the relationship between the subject, the basis of abstraction, the act-type, and the abstracted objects. Situating the discussion in mathematics education research, we contrast the proposed eidetic account with perspectives from Peirce and Piaget. Finally, we conclude by asking what implications such an account of mathematical abstraction might have for mathematics education research and practice.

Keywords: abstraction; mathematical abstraction; mathematical knowledge; eidetic epistemology; mathematics education.

1. Introduction

Acts of abstraction are widely recognized as central to the pursuit of mathematical knowledge, making their understanding crucial for promoting mathematics learning in educational contexts (e.g., Breive, 2022; Hershkowitz et al., 2001; Noss & Hoyles, 1996; Skemp, 1986; von Glasersfeld, 1991). This recognition stems from at least two reasons: First, abstraction is often invoked by mathematicians to articulate mathematics' unique nature as a knowledge domain (e.g., Dieudonné, 1955; Gowers, 2002; Weyl, 2021; Whitehead, 2017). Second, philosophers and logicians (e.g., Frege, 1960; Husserl, 2001b; Locke, 1836; Peirce, 1931–1958, 1976; Russell, 2020) and psychologists (e.g., Gelman & Gallistel, 1986; Piaget,

2014) have long been interested in the role of abstraction in mathematical activity. Drawing on these perspectives, mathematics education researchers have explored various approaches to elucidate the nature of mathematical abstraction (for overviews, see Dreyfus, 2022; Mitchelmore & White, 1995; Scheiner & Pinto, 2016).

In this paper, we aim to make a novel contribution to this longstanding discussion by addressing a twofold question: *How is mathematical abstraction possible, and what is its role in the pursuit of mathematical knowledge?* By asking this question, we set two focal points: first, we examine mathematical abstraction not only in and of itself but also in its *epistemic role*, that is, in terms of its contribution to the pursuit of mathematical knowledge. Thus, the quest for an account of mathematical abstraction is positioned within a broader context of developing an epistemology of mathematics.

Second, given the diverse interpretations of the title 'epistemology of mathematics' (see Sierpinska & Lerman, 1996), we situate our approach within a *particular* epistemological tradition, beginning with Immanuel Kant and developed by Edmund Husserl, that has so far received little attention in mathematics education research: While much work addresses how abstraction *actually* occurs in educational contexts through a *psychological* lens (e.g., Dubinsky, 1991; Gray & Tall, 1994; Mitchelmore & White, 2000; Scheiner, 2016; Sfard, 1991), we believe that investigating the *possibility* of mathematical abstraction requires a different approach, which we refer to, following Husserl (1982), as *eidetic epistemology*.[1]

The structure of the paper follows from these two focal points. First, we clarify the fundamental problem of an eidetic epistemology of mathematics, developing a specific reading of epistemological 'how-possible' questions (Section 2). The main part of the paper is then devoted to an analysis of the epistemic role and nature of mathematical abstraction (Sections 3 and 4). Finally, we situate this analysis within mathematics education research, contrasting it with perspectives from Peirce and Piaget (Section 5), and discuss its implications for mathematics education research and practice (Section 6).

2. The fundamental problem of an eidetic epistemology of mathematics

The fundamental problem of an eidetic epistemology of mathematics lies in a particular reading of the question of how mathematical knowledge is possible. This style of questioning—asking 'How is X possible?'—originates from Immanuel Kant (1724–1804), who posed the question of how pure mathematics and pure natural science are possible in his *Critique of Pure Reason* (Kant, 1998, B 20). Kant continued this line of inquiry in later works, such as the *Groundwork of the Metaphysics of Morals*, where he identified the

[1] The adjective 'eidetic', derived from the Greek noun εἶδος (eîdos), was introduced by Edmund Husserl to provide a "terminologically unspoiled name" (Husserl, 1982, p. XXVI) for characterizing research concerned with revealing the ideal conditions of possibility of any given object.

question of how a categorical imperative is possible as central to moral philosophy (Kant, 2006, 4:419–4:420).

While these historical observations provide context, they do not clarify the meaning of epistemological 'how-possible' questions. For this, we draw on an analysis provided by Edmund Husserl (1859–1938) in his *Logical Investigations*, where he distinguished between the *ideal* and *real* conditions that make human knowledge possible, and importantly, further divided the ideal conditions into *subjective* and *objective* conditions (Husserl, 2001a, p. 149–150). Figure 1 shows Husserl's typology of the conditions of possibility of human knowledge.

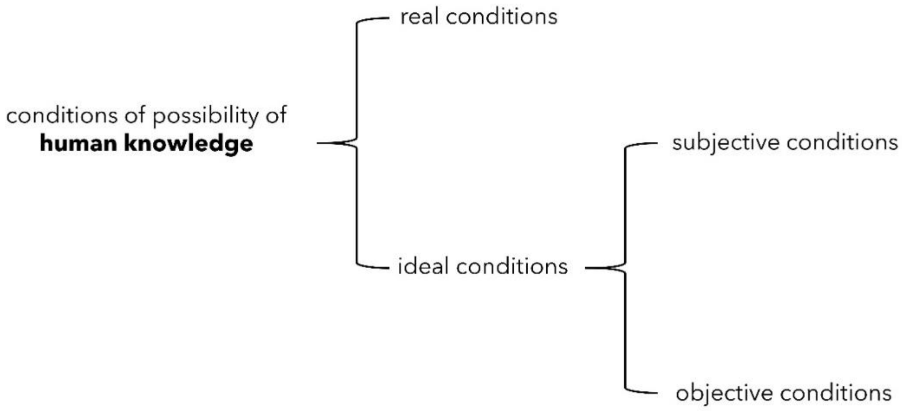

Figure 1 Husserl's typology of the conditions of possibility of human knowledge

2.1 Real and ideal conditions

The suggestion that the possibility of human knowledge depends on certain *real* conditions is not particularly surprising. For example, for a human being to come to know that 2 is an algebraic number[2], countless organic, mental, and social conditions must be fulfilled. On the organic level, the brain must be equipped with the relevant neuronal structures and supplied with sufficient oxygen; on the mental level, certain motivational dispositions must have developed; and since these dispositions often originate from participating in social processes, there are also conditions that lie on the social level.

In addition to these real conditions, however, there are also *ideal* conditions that are not bound to the "empirical peculiarity of human knowledge" (Husserl, 2001a, p. 150). A violation of these conditions would make an act of knowledge impossible, not only for a *human* being but for *any rational being*. While the real conditions are investigated by empirical disciplines such as sociology, psychology, and biology, it is *eidetic epistemology* that is concerned with uncovering the *ideal conditions of the possibility of knowledge*.

To identify such ideal conditions, the eidetic epistemologist must conduct thought experiments in which they systematically vary the conditions of the act in question. In this process of variation, it may then turn out that some conditions are tied to the specific

[2] In the context of this example, we will use the term 'algebraic number' to refer to any number that is a root of a non-zero polynomial in one variable with rational coefficients.

constitution of human beings but can be varied by imagining other rational beings, while other conditions cannot be varied without the act of knowledge ceasing to be possible.

For instance, it is possible to imagine a rational being that, unlike humans, does not need any motivating forces to come to know that 2 is an algebraic number or to think of a rational being whose sensory organs are of a different kind and that therefore expresses the relevant mathematical concepts—the nominal concept '2' and the predicative concept 'algebraic number'—by linguistic means that are quite different from those we are acquainted with. What seems impossible, however, is to imagine a rational being that comes to know that 2 is an algebraic number, even though it has not acquired the relevant concepts. If this being, for instance, does not know what it means for a number to be algebraic, then it will never be able to come to know that 2 is an algebraic number. Mastering the concepts that make up a given mathematical proposition seems to be an ideal condition for knowing the truth or falsity of that proposition.

However, it is not only an ideal condition, but also a *subjective* one, since it refers to a capacity to perform certain subjective acts. Thus, among the ideal conditions of the possibility of human knowledge, there are always conditions "whose roots lie in the form of subjectivity as such, and in its relation to knowledge" (Husserl, 2001a, p. 76). But the realm of ideal conditions does not end there. Rather, there are also *objective* conditions that make knowledge possible—"conditions which have their 'pure' foundation in the 'content' of knowledge" (Husserl, 2001a, p. 150).

2.2 Subjective and objective conditions

Mathematical knowledge has both a subjective and an objective dimension. Subjectively, it is an act performed by some rational being. Objectively, it is directed at mathematical objects, about which this subject comes to know something. In this process, the subject builds concepts, comes up with propositions and theories about the objects in question, and obtains knowledge by establishing the validity of these diverse constructions of meaning. And since these constructions of meaning—that is, the concepts, propositions, and theories—"do not hold in so far as we have insight into them, but we can only have insight into them in so far as they hold, they must be regarded as *objective* […] conditions of the possibility of our knowledge of them" (Husserl, 2001a, p. 150, italics in original). In this sense, for example, it is impossible to know that π is algebraic because, as Ferdinand von Lindemann (1852–1939) proved, π is transcendental (von Lindemann, 1882). The falsity of the proposition 'π is an algebraic number' thus functions as an objective condition for all possible acts of knowledge.

Moreover, since numerous logical conditions must be fulfilled for truth to be possible, these can also be interpreted as objective conditions for the possibility of knowledge. For example, the truth of a proposition presupposes its consistency, which is why the inconsistency of a proposition makes it impossible to come to know its truth (Husserl, 1969, p. 66). This logical insight is utilized in mathematics whenever a proof by contradiction is conducted.

Consider the following example: Can there be a number that is the result of dividing 2 by 0, if the usual laws of arithmetic apply to operations with this number, and division is interpreted as the inverse operation of multiplication? No, for if we denote this purported number by '∞', then by applying the distributive law we obtain:

$$2 = 0 \cdot \infty = (0+0) \cdot \infty = (0 \cdot \infty) + (0 \cdot \infty) = 2 + 2 = 4$$

Since the purported number is determined in an inconsistent way under these conditions, it cannot exist. These considerations highlight the importance of logical insights for an eidetic epistemology of mathematics: Whatever makes concepts, truths, inferences, or theories possible in terms of their logical form can be interpreted as an objective condition for the ideal possibility of mathematical knowledge.

In summary, an *eidetic epistemology of mathematics* is not concerned with the various *real* conditions of mathematical knowledge in humans, but rather with the *ideal* conditions that make mathematical knowledge possible not only for human beings but rather for any rational being.[3] These ideal conditions can be divided into *subjective* and *objective* conditions, the former being grounded in the acts of mathematical knowledge in their relation to the subject, and the latter in the 'objective correlates' of these acts.

3. On the epistemic role of mathematical abstraction

Having set out some important methodological considerations underlying our project of an eidetic epistemology of mathematics, we now examine where mathematical abstraction enters the scene in this project and therefore turn to the question of the *epistemic role* of mathematical abstraction. To approach this question, we begin with a reconstruction of what might be called 'the experience of the mathematician'.

3.1 The experience of the mathematician

The mathematician, broadly understood to include the early school child as well as the cutting-edge researcher, 'lives' in a mathematical attitude and thus directs their interest towards the objects of a mathematical world. Although these worlds can differ considerably depending on the level of mathematical sophistication (Tall, 2013; Lensing, 2024), they agree in that they consist of the totality of all objects whose existence can be accounted for by acts of mathematical knowledge of the respective level.

The mathematician's task is to ask questions about the objects of this mathematical world, formulate hypotheses about them, and, by pursuing these questions and hypotheses, gain knowledge about the way things are in this world. For example, the mathematician can turn to the field of natural numbers and ask whether there is a largest prime number, examine Euclidean space for the existence of a regular octahedron, or ask whether an algebraically closed ordered field exists. Here, they encounter a complex interplay of *freedom* and *boundedness*:

[3] The difference between these two types of conditions—real and ideal—does not lie in the fact that the ideal conditions are without connection to the human sphere. Every human being must, if they want to achieve mathematical knowledge, satisfy the ideal conditions. Since every conceivable rational being must fulfill these conditions, they naturally also bind all human beings. But humans must also, insofar as they are not merely rational beings, fulfill all sorts of real conditions that are not necessarily binding on other rational beings.

On the one hand, they are free to hypothesize the existence of all sorts of mathematical objects. For example, they may hypothesize that there is a largest prime number, a regular octahedron, or an algebraically closed ordered field—or equally that such objects do not exist. On the other hand, as soon as they follow the meanings that constitute these hypothetical propositions and attempt to establish the existence of the intended objects, they find that their freedom is limited by the resistance that the 'giving acts' in the mathematical sphere put up against the merely 'meaning' or 'intending acts' that determine the objects in logical freedom.

The mathematician may well *mean* that there is a largest prime number or an algebraically closed ordered field, but they cannot *know* that they exist, since it is impossible for the purported existence of these objects to be *given* to them.

3.2 The epistemic role of mathematical abstraction

These general observations lead back to the fundamental problem of an eidetic epistemology of mathematics: How is mathematical knowledge (and, correlatively, existence) possible? What are the fundamental acts in the sphere of mathematics that 'give' these objects and thus ensure their existence? It is in this context that *mathematical abstraction* comes into play.

In the following, we show that acts of mathematical abstraction often 'give' or 'present' the objects that mathematical propositions deal with *as themselves* and thus provide an important basis for any well-founded distinction between existence and non-existence, truth and falsity, and knowledge and fallacy in the mathematical sphere. In many areas of mathematics, answering the question of how mathematical knowledge is possible therefore requires first answering the question of *how mathematical abstraction is possible*.

4. Elements of an eidetic epistemology of mathematical abstraction

A comprehensive analysis of the ideal possibility of mathematical abstraction would require the scope of a book rather than a single paper. Therefore, we limit ourselves to presenting some initial elements of such an analysis. Consider the following two examples as a starting point:

> *Example 1*. Georg Cantor forms set A containing the three cities Berlin, New York, and Paris, and set B containing the three planets Jupiter, Mars, and Saturn, and then abstracts the cardinal number 3 from these sets.
>
> *Example 2*. Abraham Fraenkel introduces an ordering relation on the set A by sorting the three cities according to their population size in ascending order, then calls the ordered set A*, and abstracts the ordinal number 3 from this set.

To obtain a first description of the mathematical abstractions involved in these complex activities, we propose to distinguish between four elements:

(1) the *subject* that performs the act of abstraction (here: Georg Cantor or Abraham Fraenkel);
(2) the *act-type of abstraction* (here: the act-type that gives the cardinal or ordinal number 3);

(3) the *basis of abstraction*, that is, the objects upon which the act is carried out (here: the sets A and B or the ordered set A*); and
(4) the *abstracted object*, that is, the object that is given by the act of abstraction (here: the cardinal or ordinal number 3).

From an eidetic standpoint, the central question is now: *How can the subject, by performing an act of abstraction of a certain type, obtain the abstracted objects from the basis of abstraction?*

4.1 On the relation between the basis of abstraction and the abstracted objects

A first point to clarify is that neither the specification of the basis of abstraction nor of the abstracted objects is sufficient to determine the act of abstraction involved. Systematic variation of Example 1 illustrates this. First, the same basis of abstraction, namely the sets A and B, also offers other possibilities for abstracting. For example, by a different act of abstraction, starting from the sets A and B, one could arrive at: (a) the general type of a set whose elements are cities; or (b) at an even higher level of generality, the general type of a set whose elements are real objects.

Second, it is also evident that it is impossible to draw exact conclusions from the abstracted objects to the basis of abstraction. The sets C = {Marseille, Cologne, Barcelona} and D = {Empire State Building, Amazon River, Britney Spears} would also be a viable basis for abstracting the cardinal number 3. Thus, in mathematical abstraction: (a) the same objects can be the basis for very different abstracted objects, and (b) conversely, the same abstracted objects can originate from different bases of abstraction.

Another conclusion that can be drawn from the examples is that the basis of the abstraction can, but does not necessarily have to, consist of several objects. As illustrated in Example 2, an act of mathematical abstraction can also proceed from a single object.

4.2 On the act-types of mathematical abstraction

Another distinction, already implicitly used in Section 4.1, is even more important for an eidetic account of mathematical abstraction: the distinction between the two act-types of *generalizing* and *formalizing* abstraction (see Husserl, 1982, pp. 26–27).

Whatever mathematical object the subject is confronted with, it usually allows a ‘breakdown’ into *substance* and *form*, enabling two different *act-types* of mathematical abstraction. On the one hand, the subject can focus on the *substantial* or *material* aspects of the object and, by isolating some of them and abstracting from all others, move up in the “hierarchy of generality and specificity” (Husserl, 1982, p. 25). On the other hand, the subject can also concentrate on the *formal* aspects of the object and, by isolating some of these formal aspects and abstracting from all others as well as from any substantial or material aspects, ‘climb’ a new rung on the ladder of formality.

We illustrate this distinction with some paradigmatic examples, choosing the bases of abstraction from typical categories of mathematical objects.

For the first example, let us revisit the sets A = {Berlin, Paris, New York} and C = {Marseille, Cologne, Barcelona}. Their substantial or material aspects reside in their respective elements. The collected objects of A and C have, for instance, the material aspect in common that they are all cities or that they are all real objects. To obtain the general type of a set whose elements are cities, or of a set whose elements are real objects, acts of generalizing abstraction must be performed that isolate these common material aspects and abstract from all others.

In contrast, the act that has the sets A and C as its basis and gives the cardinal number 3 as the abstracted object is a *formalizing* abstraction, as it does not capitalize on the substantial or material aspects shared by these two sets. When it comes to the cardinality of a set, the substantial aspects of the elements are boundlessly variable: Whether the subject has collected three different cities, angels, or abelian groups into one set is a difference that makes no difference here.

To preserve the cardinality of a given set, the subject is constrained in the variation of the elements only insofar as the totality and the difference of the elements of the initial set are to be retained. Cardinality is thus a purely formal aspect of A and C which these sets, for instance, also share with the set of the divisors of 25—set E = {1, 5, 25}—or the set of abelian groups of order 8 (up to isomorphism)—set F = $\{\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\}$. To obtain the cardinal number 3 from the sets A and C, the subject has to perform a formalizing abstraction to isolate the *definite set-form* these two sets share as a purely formal aspect. This definite set-form is isolated by transforming the elements into 'units', that is, by conceiving them as placeholders for anything whatsoever and thus abstracting from all their substantial aspects.

Once this formalizing abstraction has completed its task and the cardinal numbers have been obtained as new mathematical objects, they can in turn function as a basis for further abstractions. The cardinal numbers 1, 3, and 5, for instance, can serve as the basis for a generalizing abstraction that yields the general type of an odd cardinal number. Cardinal numbers are definite set-forms that, as we saw above, consist of 'units' as their substance—that is, of items that are conceived as placeholders for anything whatsoever. These substantial aspects, by which each cardinal number can be distinguished from every other, are now not irrelevant to the abstraction in question. On the contrary, it is this substantial constitution of the cardinal numbers 1, 3, and 5—or more precisely, a certain group-ability of their units—that, if isolated by an act of generalizing abstraction, allows for the general type of an odd cardinal number to be obtained (see **Figure 2**).

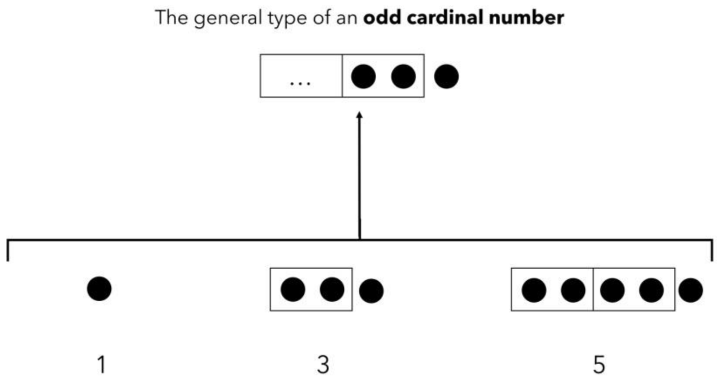

Figure 2 Representation of an act of generalizing abstraction that gives the general type of an *odd cardinal number*

However, generalizing abstraction can go even further and give more general types of cardinal numbers. This can be elucidated by expanding on the example and taking a closer look at the 'modularity' of cardinal numbers: Each cardinal number has its particular modularity—it allows for particular groupings of its units but not for others. In this context, the general type of an odd cardinal number becomes just one special case. For one can not only study the question of whether the units of a cardinal number can be grouped into 2-unit modules but also investigate their behavior when grouped into 3-, 4- or n-unit modules. Given a particular module, different cardinal numbers can thus be compared in this respect.

For example, the cardinal numbers 3 and 5 coincide in their modularity with respect to the module 2, while they differ with respect to the module 3, since 3 allows for such a grouping, while in the case of 5, two units remain ungrouped. The cardinal numbers 2, 5, and 8 behave similarly in this respect: Two units remain ungrouped in each case. If a subject now does not focus on how many 3-unit modules can be formed from the units of a particular cardinal number, but only on how many units remain in each case, then—through a generalizing abstraction—they can abstract the general type of a cardinal number with remainder 2 to the module 3 (see Figure 3).

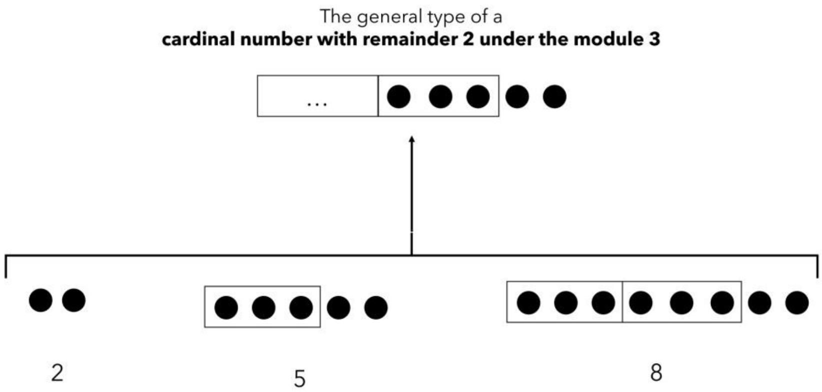

Figure 3 Representation of an act of generalizing abstraction that gives the general type of a *cardinal number with remainder 2 to the module 3*.

And the subject, then, may recognize that every third cardinal number is of this general type, that admissible arithmetic operations that are performed on two cardinal numbers of this type always yield a cardinal number of the same modularity with respect to the module 3, and thus find their way into modular arithmetic.

These analyses show that the distinction between substance and form is a *relative* distinction, so that what was a form at a lower level becomes the substance of further mathematical formations and abstractions at the next higher level, and so on.

As a consequence, higher-level mathematical objects, such as functions that map numbers to numbers, also provide a viable basis for formalizing as well as generalizing abstractions. Imagine a subject dealing with the three real-valued functions f, g, and h of a real variable x, given by the terms $f(x) = 2x + 1$, $g(x) = -\sqrt{3}x + \sqrt{2}$, and $h(x) = sin(x)$. The substantial aspects of these functions lie in their domains and codomains, as well as in the respective ways in which elements of the codomain are assigned to the elements of the domain. In these aspects of substance, there are certain commonalities that can serve the subject as a basis for different acts of generalizing abstraction.

By an act of generalizing abstraction, the subject can, for instance, obtain the general type of a linear function with real coefficients m and n from the functions f and g. Or it can, with f, g and h as a basis, and by a different act of generalizing abstraction, obtain the general type of a real-valued function of a real variable.

The subject, however, can also detach themselves from all substantial aspects and only adhere to the formal commonality that, in each case, exactly one element of a set Y is assigned to each element of a set X. By isolating this formal aspect, which manifests itself in each of the three cases but is also found in countless other cases unrelated to real numbers, the subject thus arrives at the mere form of a function through an act of formalizing abstraction. The fact that a different act-type of abstraction is needed for this could be one reason why it took so long in the history of mathematics to arrive at this formal concept of function (Kleiner, 1989).

Finally, we would like to show that the interplay between formalizing and generalizing abstraction is also at work in advanced mathematics:

> *Example 3*. Emmy Noether considers the domain of the integers $\mathbb{Z}$ together with the usual operations of addition and multiplication and then abstracts from it the ring structure $(R, +, \times)$.

To move from the domain of integers equipped with the usual operations of addition and multiplication to the ring structure $(R, +, \times)$, Emmy Noether has to perform an act of formalizing abstraction, in which the numbers and numerical operations are transformed into ‘elements’ and ‘operational forms’. The original *substance* of the elements and operations—that they were *numbers* and *number* operations—is thereby abandoned. What is retained is only a certain *form* of a mathematical domain. The domain of integers thus becomes a *domain-form* consisting of elements for which all that is known is that two binary operations are defined on them that fulfill axioms of a certain form—commonly called the ‘ring axioms’.

In this spirit, it is also possible to reduce the congruence relation from modular arithmetic to its pure relational form and thus—through an act of formalizing abstraction—elevate it to the realm of abstract algebra. For instance, a congruence relation on a ring is conceived of as an equivalence relation that is compatible with the two ring operations, so that no conditions are placed on the substance of the elements and operations in question.

However, once the ring structure has been obtained, it naturally becomes part of an ascending series of types obtained by acts of generalizing abstraction. For example, the ring structure is subordinate to the group structure, which in turn is subordinate to the magma structure, which can be regarded as the type of highest generality in the context of abstract algebra. Analogously, one can—through generalizing abstractions—obtain congruence relations of higher generality by abandoning the attachment to the ring structure and allowing attachments to groups or magmas.

4.3 On the relation between the acts of abstraction and the subject

A first important point is that we can vary the subjects in the examples without the mathematical abstractions ceasing to be possible. Any being that agrees with Cantor, Fraenkel, or Noether in that it possesses the ability to perform the necessary acts can take their place as the abstracting subject.

Another relevant aspect concerns the relation between the abstracted objects and the subject performing the act. If we compare *ideal* objects, such as the abstracted objects in the examples above, with *real* objects, such as physical things or social events, we can first point out a fundamental similarity. In both cases, 'to exist' means to be "capable of being apprehended *originaliter* as itself" (Husserl, 1973, p. 250, italics in original). Just as a tree in a meadow is given to us *originaliter* as itself through a particular act of perception, so the cardinal number 3 is given to us *originaliter* as itself through a particular act of abstraction.

However, these obvious similarities should not obscure the fundamental difference:

> [F]or every sensuous object […] its being apprehended is nonessential. The 'behavior' of the ego which motivates the procession of the multiplicities of sense data can be completely involuntary; the processes of appearance passively combine into unities in just the same way whether or not the ego turns toward what appears in them in receptive apprehension. (Husserl, 1973, p. 251)

By contrast, an abstracted object "can essentially be constituted only in a […] productive activity [that is, in an act of abstraction], [and] therefore, under the condition of the being-there ['Dabeisein'] of the ego" (Husserl, 1973, p. 251). It is thus an ideal condition of possibility for the 'presence' of abstracted mathematical objects that the subject is 'being-there' and engaged in a 'productive activity'. Mathematical objects can never be given to us as themselves in a purely receptive and involuntary way. While we hear a loud bang, for example, which captures our attention, and then see that there has been a car accident, it does not accidentally happen to us that we come to 'see' the cardinal number 3 or the ring structure. Rather, the self-givenness of such objects always involves a voluntary component.

4.4 On the ideal conditions of possibility of mathematical abstraction

We now summarize how our previous considerations can be used to address the question of the ideal conditions of possibility of mathematical abstraction. In Section 2, we distinguished between subjective and objective conditions. Accordingly, we have examined both the *objective* dimension—namely the ideal conditions relating to (a) the basis of abstraction and (b) the abstracted objects—as well as the *subjective* dimension—namely the ideal conditions relating to (c) the act-type of abstraction and (d) the subject performing the act.

Our investigations have shown that a substantiated decision of the question on whether or not a particular mathematical abstraction is ideally possible can only be made if these four elements are specified. As shown in Section 4.1, it is neither sufficient to specify the basis of abstraction nor to determine the abstracted objects to guarantee the ideal possibility of a particular mathematical abstraction. Rather, both must be determined, and the basis of abstraction must possess the relevant substantial or formal aspect to be abstracted.

For instance, it is impossible to abstract the ordinal number 3 from the set A = {Berlin, New York, Paris}, because obtaining the ordinal number 3 as an abstracted object requires an *ordered* set as the basis of abstraction. But even if, as shown in Example 2, such an ordered set A* were to emerge by an act of ordering, so that a viable basis for such an act of abstraction now exists, its ideal possibility might still falter.

The wrong act-type may be chosen (see Section 4.2), or the subject may lack the necessary abilities (see Section 4.3). For instance, to obtain the ordinal number 3 from the ordered set A*, a formalizing abstraction must be carried out, while a generalizing abstraction would ultimately fail. And even if the right act-type is chosen, the ideal possibility can still be undermined if the subject does not possess the relevant ordering abilities: For without the ability to order, no ordered sets can be constituted, and thus a viable basis upon which the desired act of formalizing abstraction could be performed is missing.

In short, the possibility of a particular mathematical abstraction is always subject to ideal conditions concerning these four elements: the basis of abstraction, the abstracted objects, the act-type of abstraction, and the subject. *The ideal possibility is only assured if these four elements are adequately related to each other*.

5. Situating our eidetic account in mathematics education research

Now that we have presented elements of an eidetic account of mathematical abstraction, we compare this account with two influential strands in mathematics education research.

5.1 Charles Sanders Peirce and the notion of hypostatic abstraction

One strand of research on mathematical abstraction consists of approaches that build on the work of Charles Sanders Peirce (1839–1914) (e.g., Bakker, 2007; Bakker & Hoffmann, 2005; Hoffmann, 2005). Throughout his professional life, Peirce grappled with the puzzles that the existence of mathematics poses to epistemology, such as how to make the “striking characteristics of mathematics” (Peirce, 1931–1958, CP 4.237) transparent. Where does “the fleshless and skeletal build of its propositions; the peculiar difficulty, complication, and stress of its reasonings; the perfect exactitude of its results; their broad universality; their practical infallibility" (Peirce, 1931–1958, CP 4.237) come from?

In exploring these questions, Peirce arrived at a semiotic account of mathematical knowledge (see Otte, 2006). Among other things, he emphasized the mediating role of signs—particularly that of diagrams—in mathematical reasoning (Peirce, 1931–1958, CP 5.148), stressed the importance of imagination and thought experiments in the mathematical sphere, and argued for the crucial epistemic role of the "act of hypostatic abstraction" (Peirce, 1931–1958, CP 5.455), which he described as "an essential part of almost every really helpful step in mathematics" (Peirce, 1976, IV, p. 160).

Although a detailed discussion of Peirce's semiotics and epistemology of mathematics is beyond the scope of this paper, we would at least like to compare our account of mathematical abstraction with Peirce's account of *hypostatic abstraction*.

> In order to get an inkling—though a very slight one—of the importance of this operation in mathematics, it will suffice to remember that a *collection* is a hypostatic abstraction, or *ens rationis,* that *multitude* is the hypostatic abstraction derived from a predicate of a collection, and that a *cardinal number* is an abstraction attached to a multitude. (Peirce, 1931–1958, CP 5.534, italics in original)

In our analyses, we agree with Peirce that acts of mathematical abstraction are 'giving acts', in which the "creation of *ens rationis*" (Peirce, 1931–1958, CP 5.449, italics in original) is accomplished. Just as Peirce argued against the view that an act of mathematical abstraction can be adequately understood as a "delicate shift of attention" (Mason, 1989, p. 2) because nothing new can arise from such shifts of attention (Peirce, 1931–1958, CP 4.235), so we have argued in favor of the 'creational power' of acts of mathematical abstraction.

Moreover, Peirce's emphasis that acts of mathematical abstraction are usually accompanied by an act of *hypostatization* or *objectification*, which turns the abstracted into an object available for further mathematical activity, adds something to our eidetic account.[4] For example, we have argued that the cardinal number 3 is given through an act of formalizing abstraction. Starting from the sets A and B, the cardinal number 3 was obtained as an abstracted object by first transforming the respective elements of the two sets into completely arbitrary items, so that only their totality and difference remained—to then isolate the definite set-form common to both sets as a purely formal aspect. Here, an act of *hypostatization* or *objectification* is involved, making this form itself an object for further investigation.

However, alongside these synergies, there are also some crucial differences. In particular, we do not believe, as Peirce seems to have suggested, that the involvement of this act of hypostatization fully captures the determining characteristics of acts of mathematical abstraction. To clarify this point of deviation, we analyze the examples Peirce provided in the quoted passage.

[4] For different accounts of objectification in mathematics education research, see Radford (2021), Sfard and Linchevski (1994), and Tall et al. (1999).

Peirce seems to have identified the acts by which *collections* or *sets* are given to us with the acts by which *cardinal numbers* are given to us, conceiving both as acts of hypostatic abstraction. From our perspective, however, this identification cannot be maintained. We have already examined how the cardinal number 3 is given through an act of formalizing abstraction, which involves an objectification or hypostatization of the definite set-form. Otherwise, the cardinal number 3 could neither be examined for its properties nor investigated for its numerical relationships. But it seems to us that the actual core of the activity, namely the step of formalization, is missed if one says that *only* such an act is necessary.

It is now enlightening to compare *this* type of giving act, in which the cardinal number 3 is obtained, with those acts by which the *collection* or *set* A = {Berlin, New York, Paris} is given. This collection or set, we argue, does not result from an act of abstraction but rather from an "act of colligation" (Husserl, 1973, p. 246), followed by an act of objectification. Initially, an act of colligation is performed, linking the objects—Berlin, New York, and Paris—through the form of collective connection. Here, "the colligating consciousness contains several objects encompassed in unity but not a unique object having several members. Nevertheless, […] a new object is essentially preconstituted, precisely one having many members," the set {Berlin, New York, Paris}, which, after the act of colligation is completed, can be objectified and thus itself become the "substrate of [further] judgment" (Husserl, 1973, p. 246). Thus, although we are dealing with the transition that Peirce describes as 'hypostatic abstraction' *after* the act of colligation is completed, we argue that this transition *cannot* be considered the essence of the mathematical activity by which the set A is constituted.

Of course, one still grasps an important aspect of the respective activities by which sets and cardinal numbers are given, if one puts the focus—as Peirce did—on the involved acts of *hypostatization* or *objectification*. However, since our analysis also exhibits fundamental differences between the respective activities that give sets or cardinal numbers, we believe that their description loses some of its precision when these two acts are identified.

5.2 Jean Piaget and the notion of reflective abstraction

Another influential strand of research on mathematical abstraction draws on the work of Jean Piaget (1896–1980) (e.g., Cottrill et al., 1996; Dubinsky, 1991; Simon et al., 2004; Steffe, 1991; von Glasersfeld, 1995). Piaget notably distinguished between two basic types of abstraction, *empirical* and *reflective* abstraction. While he held that empirical abstraction can be considered central to the formation of empirical knowledge, he argued that reflective abstraction is crucial to the acquisition of logical and mathematical knowledge.

> Let us ask what logical and mathematical knowledge is abstracted from. There are two possibilities. The first is that, when we act upon an object, our knowledge is derived from the object itself. This is the point of view of empiricism in general, and it is valid in the case of experimental or empirical knowledge for the most part. But there is a second possibility: when we are acting upon an object, we can also take into account the action itself, or operation if you will, since the transformation can be carried out mentally. In this hypothesis the abstraction is drawn not from the

> object that is acted upon, but from the action itself. It seems to me that this is the basis of logical and mathematical abstraction. (Piaget, 1971, p. 16)

Piaget's formulations suggest that knowledge, whether empirical or logical and mathematical, can result from acts of abstraction. From the perspective of an eidetic epistemology, we object to this view, as acts of abstraction are neither sufficient nor necessary for coming to know something in mathematics. As shown in Section 3, acts of abstraction cannot be *sufficient* to obtain mathematical knowledge, for in an act of knowledge, always two aspects must be brought to correspondence: (a) a meaning or intending act and (b) a giving act. For instance, to come to know that 3 is a prime number, a subject must examine the number for its divisors, and for this purpose, the number must be given to them, meaning that an act of abstraction must be embedded in the act of knowledge. This does not mean, however, that the knowledge emerging from such an act can itself be regarded as the result of the abstraction involved. What is missing here is the meaning or intending act, so that acts of abstraction can never make up a full act of knowledge but can only be a relevant part of it.

However, they must not necessarily be involved for mathematical knowledge to be possible, as there are also cases where the objects about which the subject comes to know something are not given to them by acts of abstraction. In Section 5.1, we showed that sets are given as themselves by acts of colligation, so that it is possible to know something about sets without any abstraction involved; but one can also think, for example, of the ideal figures of Euclidean geometry, which are given to us not by abstraction but by acts of idealization (Husserl, 1982, pp. 166–167).

Another important question concerns how Piaget's distinction between empirical and reflective abstraction relates to our distinction between generalizing and formalizing abstraction. To respond to this query, it must first be clarified what Piaget means by empirical and reflective abstraction, an issue that has generated extensive debate in mathematics education research (e.g., Dubinsky, 1991; Ellis et al., 2024; Goodson-Espy, 1998; Simon et al., 2004; Tall, 2013; von Glasersfeld, 1995).

In our view, there are two approaches to this issue: On the one hand, researchers in mathematics education often rely on Piaget's well-known quotations that succinctly express the difference (e.g., Piaget, 2014, p. 317). On the other hand, a closer examination of his actual analyses—for instance, on the development of the conception of number (e.g., Piaget, 1973, p. 75)—could also form the basis for a reconstruction of the distinction. We expect these two approaches to yield different results, with the latter potentially bringing Piaget's account much closer to the views we advocate here, following Husserl, than the former.

However, a detailed reconstruction is beyond the scope of this paper; thus, we limit our discussion to one influential interpretation of 'reflective abstraction'. This interpretation suggests that mathematical objects and concepts can be obtained by reflecting on the mental acts or operations through which these objects or concepts are given, thus making these mental activities the basis of abstraction. This interpretation of reflective abstraction is widely recognized in mathematics education research. For example, von Glasersfeld (1995) explained that "the material *from* which these [reflective] abstractions are formed, consists of operations that the thinking subject performs and reflects upon" (p. 69, italics in original), a view also held by Dubinsky (1991, p. 97). Against this, with Husserl, we argue:

> It is a natural but quite misguided doctrine, universally put about since the time of Locke, that […] *categories* such as being and non-being, unity, plurality, totality, number, ground, consequence etc.—arise through *reflection upon certain mental acts, and so fall in the sphere of 'inner sense', of 'inner perception'*. In this manner, indeed, concepts like Perception, Judgement, Affirmation, Denial, Collecting, Counting, Presupposing and Inferring arise, which are all, therefore, 'sensuous' concepts, belonging […] to the sphere of 'inner sense'. The previous series of concepts do not arise in this manner, since they cannot at all be regarded as concepts of mental acts, or of their real constituents. (Husserl, 2001b, p. 278, italics in original)

For example, when we collect the cities Berlin, New York, and Paris, then objectify the set A = {Berlin, New York, Paris} pre-constituted in this act of colligating, and finally, by performing an act of formalizing abstraction, obtain the cardinal number 3, our focus remains objectively oriented throughout. We carry out various mental acts (e.g., colligating, objectifying, abstracting) through which the respective objects (e.g., the set A or the cardinal number 3) are given. But we are not directed towards these acts themselves, but to the corresponding objects. As our analyses have shown, in the mathematical activity through which the cardinal number 3 is given to us as itself, there is no reflection on our mental acts at any point.

6. Conclusion

In this paper, we have embarked on the project of unraveling some ideal conditions of the possibility of mathematical abstraction, situating this undertaking as part of a broader effort to develop an *eidetic epistemology of mathematical knowledge*. To conclude this paper, we make a few remarks about what implications an eidetic account of mathematical abstraction might have for mathematics education research and practice.

Eidetic research into the possibility of mathematical abstraction differs fundamentally from *empirical* research. It does not concern itself with the real conditions that make mathematical abstraction possible for particular groups of human beings (e.g., students of a particular age in a particular societal context) or those tied to the specific constitution of the human species. Instead, eidetic research seeks to uncover the ideal conditions that human beings must also adhere to, since they make mathematical abstraction possible for any rational being. Despite this distinction, by revealing such ideal conditions, eidetic research can provide an interpretative framework for the empirical study of mathematical abstraction.

For instance, if one investigates the actual attempts of primary school children to perform acts of abstraction that give cardinal and ordinal numbers, an eidetic account allows for a more precise analysis of what causes difficulties in the actual performance of these and why many such acts do not come to fruition. For example, one can examine whether the basis of abstraction and the objects to be abstracted fit together, whether the appropriate act-type has been chosen, or whether the subject possesses the ability to perform the acts in which the objects that form the basis of abstraction are constituted.

Beyond its relevance to research, an eidetic epistemology has also significant implications for mathematics education practice. If acts of mathematical abstraction are only possible under the ideal conditions that eidetic research attempts to uncover, then such research allows to derive normative principles that can guide the design of learning environments. For example, suppose one wonders what a learning environment would have to be like for students to learn to perform the acts of abstraction that give rise to ordinal and cardinal numbers. In that case, the eidetic analysis of the corresponding acts of formalizing abstraction provides a means of subjecting each *proposed* learning environment to critical scrutiny. Are students enabled to constitute the relevant basis of abstraction? And do the objects that form this basis match the objects to be abstracted? Are students already able to perform the respective acts of formalizing abstraction? And if not, are sufficient opportunities provided to facilitate the initial performance of such acts?

However tentative as our eidetic analysis of mathematical abstraction in this paper may be, we believe they point to an important line of inquiry with relevance to both mathematics education research and practice. By continuing to refine our understanding of the ideal conditions of mathematical abstraction and its role in the pursuit of mathematical knowledge, a stronger theoretical foundation can be laid for mathematics education research *and* practice.

References

Bakker, A. (2007). Diagrammatic reasoning and hypostatic abstraction in statistics education. *Semiotica*, *164*, 9–29. https://doi.org/10.1515/SEM.2007.017

Bakker, A., & Hoffmann, M. H. G. (2005). Diagrammatic reasoning as the basis for developing concepts: A semiotic analysis of students' learning about statistical distribution. *Educational Studies in Mathematics*, *60*(3), 333–358. https://doi.org/10.1007/s10649-005-5536-8

Breive, S. (2022). Abstraction and embodiment: exploring the process of grasping a general. *Educational Studies in Mathematics*, *110*(2), 313–329. https://doi.org/10.1007/s10649-021-10137-x

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. *Journal of Mathematical Behavior*, *15*(2), 167–192. https://doi.org/10.1016/s0732-3123(96)90015-2

Dieudonné, J. (1955). L'abstraction en mathématiques et l'évolution de l'algèbre [Abstraction in mathematics and the evolution of algebra]. In J. Piaget, E. W. Beth, J. Dieudonné, A. Lichnerowicz, G. Choquet, & C. Gattegno (Eds.), *L'enseignement des mathématiques* (pp. 47–61). Delachaux et Niestlé.

Dreyfus, T. (2022). Abstraction in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 13–16). Springer. https://doi.org/10.1007/978-94-007-4978-8_2

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. O. Tall (Ed.), *Advanced mathematical thinking*, (pp. 95–123). Kluwer. https://link.springer.com/chapter/10.1007/0-306-47203-1_7

Ellis, A., Paoletti, T., & Lockwood, E. (2024). Empirical and reflective abstraction. In P. C. Dawkins, A. J. Hackenberg, & A. Norton (Eds.), *Piaget's genetic epistemology for mathematics education research* (pp. 169–208). Springer. https://doi.org/10.1007/978-3-031-47386-9_6

Frege, G. (1960). *The foundations of arithmetic: A logico-mathematical enquiry into the concept of number*. Harper & Brothers.

Gelman, R., & Gallistel, C. R. (1986). *The child's understanding of number*. Harvard University Press. https://doi.org/10.4159/9780674037533

Goodson-Espy, T. (1998). The roles of reification and reflective abstraction in the development of abstract thought: Transitions from arithmetic to algebra. *Educational Studies in Mathematics*, *36*(3), 219–245. https://doi.org/10.1023/A:1003473509628

Gowers, T. (2002). *Mathematics: A very short introduction*. Oxford University Press. https://doi.org/10.1093/actrade/9780192853615.001.0001

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A 'proceptual' view of simple arithmetic. *Journal for Research in Mathematics Education*, *25*(2), 116–140. https://doi.org/10.5951/jresematheduc.25.2.0116

Hershkowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education*, *32*(2), 195–222. https://doi.org/10.2307/749673

Hoffmann, M. H. G. (2005). Signs as means for discoveries: Peirce and his concepts of 'diagrammatic reasoning', 'theorematic deduction', 'hypostatic abstraction', and 'theoric transformation'. In M. H. G. Hoffmann, J. Lenhard, & F. Seeger (Eds.), *Activity and sign: Grounding mathematics education* (pp. 45–56). Springer.

Husserl, E. (1969). *Formal and transcendental logic*. Martinus Nijhoff Publishers. (Original work published 1929). https://doi.org/10.1007/978-94-017-4900-8

Husserl, E. (1973). *Experience and judgment*. Northwestern University Press. (Original work published 1939).

Husserl, E. (1982). *Ideas pertaining to a pure phenomenology and to a phenomenological philosophy*. Martinus Nijhoff Publishers. (Original work published 1913).

Husserl, E. (2001a). *Logical investigations: Volume 1*. Routledge. (Original work published 1900). https://doi.org/10.4324/9780203879054

Husserl, E. (2001b). *Logical investigations: Volume 2*. Routledge. (Original work published 1901).https://doi.org/10.4324/9780203879047

Kant, I. (1998). *Critique of pure reason*. Cambridge University Press. (Original work published 1781). https://doi.org/10.1017/CBO9780511804649

Kant, I. (2006). *Groundworks of the metaphysics of morals*. Cambridge University Press. (Original work published 1785). https://doi.org/10.12987/9780300128154

Kleiner, I. (1989). Evolution of the function concept: A brief survey. *The College Mathematics Journal*, *20*(4), 282–300. https://doi.org/10.1080/07468342.1989.11973245

Lensing, F. (2024). Friedrich Drenckhahns Überlegungen zur disziplinären Identität der Mathematikdidaktik und die Idee einer Theorie der mathematischen Erkenntnisstufen [Friedrich Drenckhahn's thoughts on the disciplinary identity of mathematics education research and the idea of a theory of stages of mathematical knowledge]. In R. Bruder, A. Büchter, & R. Sträßer (Eds.). *Fallstudien zur Geschichte der Mathematikdidaktik*. (pp. 65–84). WTM.

Locke, J. (1836). *An essay concerning human understanding*. T. Tegg and Son.

Mason, J. (1989). Mathematical abstraction as the result of a delicate shift of attention. *For the Learning of Mathematics*, *9*(2), 2–8. https://www.jstor.org/stable/40247947

Mitchelmore, M. C., & White, P. (1995). Abstraction in mathematics: Conflict, resolution and application. *Mathematics Education Research Journal*, *7*(1), 50–68. https://doi.org/10.1007/BF03217275

Mitchelmore, M. C., & White, P. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics*, *41*, 209–238. https://doi.org/10.1023/A:1003927811079

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Springer. https://doi.org/10.1007/978-94-009-1696-8

Otte, M. (2006). Mathematical epistemology from a Peircean semiotic point of view. *Educational Studies in Mathematics*, *61*(1–2), 11–38. https://doi.org/10.1007/s10649-006-0082-6

Peirce, C. S. (1931–1958). *Collected papers of Charles Sanders Peirce* (Vol. 1–8, C. Hartshorne, P. Weiss, & A. Burks, Eds.). Harvard University Press.

Peirce, C. S. (1976). *The new elements of mathematics by Charles S. Peirce* (Vol. I–IV). Mouton & Co.

Piaget, J. (1971). *Genetic epistemology*. W.W. Norton & Company Inc. https://doi.org/10.7312/piag91272

Piaget, J. (1973). *Introduction à l'épistémologie génétique 1: La pensée mathématique* [Introduction to genetic epistemology 1: mathematical thought]. Presses Universitaires de France.

Piaget, J. (2014). *Studies in reflecting abstraction* (R. L. Campell, Ed. & Trans.). Psychology Press. (Original work published 1977). https://doi.org/10.4324/9781315800509

Radford, L. (2021). *The theory of objectification: A Vygotskian perspective on knowing and becoming in mathematics teaching and learning*. Brill. https://doi.org/10.1163/9789004459663

Russell, B. (2020). *Principles of mathematics*. Routledge. (Original work published 1903). https://doi.org/10.4324/9780203864760

Scheiner, T. (2016). New light on old horizon: Constructing mathematical concepts, underlying abstraction processes, and sense making strategies. *Educational Studies in Mathematics*, *91*(2), 165–183. https://doi.org/10.1007/s10649-015-9665-4

Scheiner, T., & Pinto, M. M. F. (2016). Images of abstraction in mathematics education: Contradictions, controversies, and convergences. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.), *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 155–162). PME.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1–36. https://doi.org/10.1007/BF00302715

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification—the case of algebra. *Educational Studies in Mathematics*, *26*(2–3), 191–228. https://doi.org/10.1007/BF01273663

Simon, M. A., Tzur, R., Heinz, K., & Kinzel, M. (2004). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education*, *35*(5), 305–329. https://doi.org/10.2307/30034818

Sierpinska, A., & Lerman, S. (1996). Epistemologies of mathematics and of mathematics education. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education: Part 1* (pp. 827–876). Springer. https://doi.org/10.1007/978-94-009-1465-0_23

Skemp, R (1986). *The psychology of learning mathematics* (2nd ed., first published 1971). Penguin. https://doi.org/10.4324/9780203396391

Steffe, L. P. (Ed.). (1991). *Epistemological foundations of mathematical experience*. Springer. https://doi.org/10.1007/978-1-4612-3178-3

Tall, D. O. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*. Cambridge University Press. https://doi.org/10.1017/CBO9781139565202

Tall, D. O., Thomas, M., Davis, G., Gray, E., & Simpson, A. (1999). What is the object of the encapsulation of a process? *The Journal of Mathematical Behavior*, *18*(2), 223–241. https://doi.org/10.1016/S0732-3123(99)00029-2

von Glasersfeld, E. (1991). Abstraction, re-presentation, and reflection: An interpretation of experience and Piaget's approach. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 45–67). Springer. https://doi.org/10.1007/978-1-4612-3178-3_4

von Glasersfeld, E. (1995). *Radical constructivism: A way of learning and knowing*. Routledge. https://doi.org/10.4324/9780203454220

von Lindemann, F. (1882). Über die Zahl π. *Mathematische Annalen*, *20*(2), 213–225.

Weyl, H. (2021). *Philosophy of mathematics and natural science* (O. Helmer, Trans.). Princeton University Press. https://doi.org/10.2307/j.ctv1t1kftd

Whitehead, A. N. (2017). *An introduction to mathematics*. Dover Publications. (Original work published 1911).