

INTERVIEW WITH CARLO CELLUCCI ON HIS HEURISTIC PHILOSOPHY OF MATHEMATICS

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In the following interview, Carlo Cellucci, professor emeritus of the Sapienza University of Rome, who has written books on logic, mathematics and philosophy, discusses his 2022 book, *The Making of Mathematics – Heuristic Philosophy of Mathematics*, which presents a new paradigm for thinking about, doing and teaching mathematics.

Gordon: Mathematics, of all the disciplines, has enjoyed special privilege in being recognized as that body of knowledge which develops and constitutes truth through logical argument. Yet you state that mathematical truth doesn't exist – instead, the most we can believe is that our findings are plausible. Would you share your thinking?

Cellucci: If mathematics were that body of knowledge which develops and constitutes truth through logical argument – assuming that by 'logical argument' you mean 'deductive argument' – then it would be impossible to do mathematics. For, by Gödel's incompleteness theorems, mathematicians would never be able to know whether their theorems are true or not.

Most mathematicians don't accept that Gödel's incompleteness theorems are relevant to them. Thus, Davies says: "I got the dread seeing yet another discussion of Gödel's theorems and their importance, when I knew that they had almost no relevance to the work of most mathematicians" (Davies 2008, 88). However, at least Davies mentions Gödel. On the contrary, Bourbaki even goes so far as to make no mention of Gödel. As Mathias says, in their works the Bourbachistes "ignored him, except that the tone of certain of their works suggests a conflict between an uneasy awareness that something has happened and a desire to pretend that it has not. It is as though they had discovered that they were on an island with a dragon and in response chose to believe that if the dragon were given no name it would not exist" (Mathias 1992, 6).

Despite the denialist attitude of most mathematicians, Gödel's incompleteness theorems deeply affect them. For, Gödel's results show that the way most mathematicians conceive of their work is untenable.

Nathanson says: "We mathematicians have a naive belief in truth. We prove theorems. Theorems are deductions from axioms. Each line in a proof is a simple consequence of the previous lines of the proof, or of previously proved theorems. Our conclusions are true, unconditionally and eternally" (Nathanson 2008, 773). Thus, Mac Lane says: "The traditional nature of a proof remains. It is a deduction from suitable axioms," the nature of "proof is eternal" (Mac Lane 1984, 44). Rota says: "Once solved, a mathematical problem is forever finished: no later event will disprove a correct solution" (Rota 1997, 93).

But, by Gödel's incompleteness theorems, this view of mathematics is untenable. On the one hand, by Gödel's second incompleteness theorem, it is impossible to prove, by any reliable means, that the conclusions of mathematical proofs are true. On the other hand, it is impossible to say that mathematical truths are deductions from axioms. For, by a corollary of Gödel's first incompleteness theorem, the strong incompleteness theorem for second-order logic, there is no set of deductive rules capable of deducing all the second-order consequences

of any given set of axioms. Thus, there is no set of deductive rules capable of deducing all number-theoretical truths from Peano's axioms.

Most mathematicians don't accept that Gödel's incompleteness theorems are highly relevant to them because, as Lovász says, we "mathematicians are conservative people," we "don't push for changes," so "we pretend that mathematical research is as it used to be" (Lovász 1998, 33).

Gordon: Mathematics, of all the disciplines, has also enjoyed special privilege in being recognized as that body of knowledge which provides results that last forever. Yet you state that mathematical results don't last forever. Would you share your thinking?

Cellucci: The belief that mathematical results last forever is widespread. Thus, as we have seen, Rota says that once solved, a mathematical problem is forever finished. As another example, Krantz says that, "once a theorem is proved, and its proof checked and validated, then the theorem stands forever. It is just as true, and just as useful, today as when it was proved" (Krantz 2011, 169). But this belief is unfounded. As Goodman says, "the results of mathematics are no more certain or everlasting than the results of any other science, even though, for sociological reasons, our histories of mathematics tend to disguise that fact" (Goodman 1991, 125).

The belief that mathematical results last forever is due to the assumption that, while there are revolutions in science, there are no revolutions in mathematics, so, unlike the development of science, the development of mathematics is purely cumulative, namely it simply proceeds by addition of new results to the results that are already known. But this assumption is unfounded. Thus, compare the development of calculus in the eighteenth century and in the nineteenth century. Mathematicians in the eighteenth century freely manipulated divergent series and so came to results that, with Cauchy's revolution of the nineteenth century, appeared absurd. Therefore Euler's theory of infinite series was completely abandoned. This is a significant example of the fact that the development of mathematics is not purely cumulative.

A related belief is that errors in mathematical proofs and results are detected within a reasonably short time. This belief is also unfounded. For example, one of the famous 23 Hilbert's problems posed at the Second International Congress of Mathematicians in 1900 was considered to have been solved by Dulac in 1923, and Dulac died in 1955 believing he had solved it. But a counterexample was found by Ilyashenko in 1982, about sixty years later.

Gordon: Mathematics is said to be that body of knowledge established by proving theorems. Yet you argue that mathematics is not constituted by proving theorems but by solving problems. What is the distinction you would hope would be recognized here?

Cellucci: Mathematics is currently said to be that body of knowledge established by proving theorems by the axiomatic method. But, by Gödel's incompleteness theorems, this is untenable. The current view that mathematics is theorem proving by the axiomatic method has been conclusively refuted by Gödel's incompleteness results. On the contrary, the view that mathematics is problem solving by the analytic method is compatible with Gödel's incompleteness results. In fact, it is the only view of mathematics compatible with them.

Some mathematicians are aware that mathematics is not theorem proving by axiomatic method. Thus, Hersh says: "A naive non-mathematician" who "looks into Euclid" and "observes that axioms come first," justifiably "concludes that in mathematics, axioms come first. First your assumptions, then your conclusions, no? But anyone who has done mathematics knows what comes first – a problem." Indeed, "mathematics is a vast network of

interconnected problems and solutions,” in developing and understanding a subject, “problems, and solutions come first,” therefore “the view that mathematics is in essence derivations from axioms is backward. In fact, it’s wrong” (Hersh 1997, 6).

As another example, Hamming says that “mathematics is not simply laying down some arbitrary postulates and then making deductions” from them, on the contrary, “you start with some of the things you want and you try to find postulates to support them” (Hamming 1998, 645). Thus, “Euclid’s postulates came from the Pythagorean theorem, not the other way” (Hamming 1980, 87).

The statement that Euclid’s postulates came from the Pythagorean theorem expresses the substance of the analytic method.

Gordon: How would you distinguish the axiomatic method from the analytic method?

Cellucci: As Kac and Ulam say, the axiomatic method “consists in starting with a few statements (axioms) whose truth is taken for granted and then deriving other statements from them by the application of rules of logic alone.” (Kac and Ulam 1992, 139). Since the truth of the axioms is taken for granted, no justification for the axioms is given. And since the other statements are derived, namely deduced, from the axioms, only deductive logic is required. The purpose of the axiomatic method is to justify statements by deducing them from axioms, so the axiomatic method is intended to be a method of justification. However, the justification of the statements deduced from the axioms depends on the justification of the axioms, which not only is not given, but by Gödel’s incompleteness theorems cannot be given. Therefore, no justification for the statements deduced from the axioms can be given. Thus, the whole purpose of the axiomatic method evaporates.

On the other hand the analytic method, also called the method of analysis, is the method according to which, to solve a problem, one starts from the problem and possibly other data already available, and obtains a hypothesis from them by some non-deductive rule, such as induction or analogy. Then one justifies the hypothesis by showing that the hypothesis is plausible, namely the arguments for it are stronger than the arguments against it, on the basis of the existing experience. Finally, one deduces a solution to the problem from the hypothesis. Thus, both non-deductive logic and deductive logic are required. The analytic method involves two movements, an upward movement from the problem to the hypotheses, carried out by non-deductive rules, and a downward movement from the hypotheses to the problem, carried out by deductive rules. The purpose of the analytic method is to discover a solution to a problem, so the analytic method is intended to be a method of discovery.

While, in the axiomatic method, mathematical truths are supposed to be true forever, in the analytic method, plausible hypotheses are not plausible forever. A hypothesis that is plausible at one time can become implausible at a later time, when the arguments against the hypothesis become stronger than those for it.

Gordon: You consider non-deductive logic as instrumental in problem solving. What is the difference between non-deductive logic and deductive logic?

Cellucci: Non-deductive logic consists of non-deductive rules, that is, rules that are ampliative, namely such that the conclusion is not contained in the premisses. An example of non-deductive rule is induction, the rule by which Bachet inferred, from the premiss ‘Every natural number up to 325 is either a square, or the sum of two, three, or four squares,’ which expressed a fact he had observed, the conclusion “Every number is either a square, or the sum of two, three, or four squares” (Bachet 1621, 241). Thus Bachet arrived to formulate this hypothesis about the integers. The conclusion is not contained in the premiss. Non-deductive

rules are not truth-preserving, namely such that, if the premisses are true, the conclusion is not necessarily true. This is because non-deductive rules are ampliative.

On the other hand, deductive logic consists of deductive rules, that is, rules that are non-ampliative, namely such that the conclusion is contained in the premisses. An example of deductive rule is modus ponens, the rule by which one infers, from the premisses ‘ $2+2=4$ ’ and ‘If $2+2=4$, then $2+3=5$,’ the conclusion ‘ $2+3=5$.’ The conclusion is contained in the premiss ‘If $2+2=4$, then $2+3=5$.’ Deductive rules are truth-preserving, namely such that, if the premisses are true, the conclusion is true. This is because deductive rules are non-ampliative.

Gordon: How does the difference between the view that mathematics is theorem proving by the axiomatic method and the view that mathematics is problem solving by the analytic method, affect the nature of mathematical theories?

Cellucci: The view that mathematics is theorem proving by the axiomatic method implies that mathematical theories are closed systems, namely systems whose development is completely internal to the system. For, it is based on axioms given once for all, and consists entirely in deducing propositions from the axioms. Since deductive rules are non-ampliative, all theorems are already implicitly contained in the axioms. As Hintikka says, in the axiomatic method, you compress “all the truths about” a given “subject matter into” a “set of axioms,” which “are supposed to tell you everything there is to be told about this subject matter.” Then “the rest of your work will consist in merely teasing out the logical consequences of the axioms. It is sufficient to study the axioms” (Hintikka 1998, 1).

Conversely, the view that mathematics is problem solving by the analytic method implies that mathematical theories are open systems, namely systems whose development takes place by interacting with other systems, receiving inputs from them and delivering outputs to them. Since non-deductive rules are ampliative, the hypotheses are not already contained in the problem, and need not belong to the same part of mathematics as the problem, they may belong to other parts of mathematics.

Gordon: What are the advantages of the view that mathematics is problem solving by the analytic method over the view that mathematics is theorem proving by the axiomatic method?

Cellucci: The fundamental advantage is that, as already said, while the view that mathematics is theorem proving by the axiomatic method has been conclusively refuted by Gödel’s incompleteness theorems, the view that mathematics is problem solving by the analytic method is compatible with them, and in fact is the only view compatible with them. But there are also several other advantages.

For example, the view that mathematics is theorem proving by the axiomatic method does not account for the fact that solving a problem of a given part of mathematics may require hypotheses from other parts of mathematics. Thus, to solve the problem posed by Fermat’s conjecture, which is a problem about the integers, Ribet used a hypothesis about modular forms in hyperbolic space, the Taniyama-Shimura conjecture: Every elliptic curve over the rational numbers is modular. Then, to solve the problem posed by the Taniyama-Shimura conjecture, Wiles and Taylor used hypotheses from various parts of mathematics, from differential geometry to complex analysis. The view that mathematics is theorem proving by the axiomatic method does not account for this fact, because it demands that a solution to a problem of a given part of mathematics be deduced from the axioms of that part. Hilbert called this demand “the demand for ‘purity’ of method” (Hilbert 1987, 125). On the contrary, the view that mathematics is problem solving by the analytic method accounts for this fact because, as already said, in the analytic method the hypotheses to solve a problem need not

belong to the same part of mathematics as the problem, they may belong to other parts of mathematics.

As another example, the view that mathematics is theorem proving by the axiomatic method does not account for the fact that new solutions are often sought for problems for which a solution is already known. Thus, for the Pythagorean theorem, *over four hundred proofs are known, and their number is still growing*. The view that mathematics is theorem proving by the axiomatic method does not account for this fact because, according to it, a proof establishes the truth of a theorem, and this is its function. Then, once the truth of a theorem has been established, there is no point in establishing it once again by another proof, let alone by hundreds of them. On the contrary, the view that mathematics is problem solving by the analytic method accounts for this fact because, according to it, each new solution establishes new connections between the problem and the existing knowledge, including other parts of mathematics, so each new solution can lead to progress even when a solution is already known.

Gordon: Mathematical discovery has historically been considered illogical, idiosyncratic, intuitive – mysterious. Indeed, the philosopher Karl Popper made the argument that the ‘context of discovery’ is so different from the ‘context of justification’ that the former should not even be discussed in the context of presenting formal arguments. Yet you argue that there is a ‘logic of discovery’ and that ‘discovery is objective.’ Please comment.

Cellucci: Popper is inconsistent because, on the one hand, he claims that there is no method of discovery, in particular, induction does not exist, on the other hand, he claims that solutions to problems are discovered by the method of trial and error, which is an inductive method because, as Popper admits, the success of trials “depends very largely on the number and variety of the trials: the more we try, the more likely it is that one of our attempts will be successful” (Popper 1974, 312). What is more important, Popper’s claim that solutions to problems are discovered by the method of trial and error is untenable because the number of trials mathematicians can make is very small compared to all possible ones, so the probability that a single trial will be successful is very low. This contrasts with the fact that very many mathematics research articles are published every year, for example, there were “104,953 journal articles published in 2017” (Dunne 2019, 227). This would be inexplicable if mathematicians discovered solutions to problems by trial and error.

As to the view that mathematical discovery has historically been considered illogical, this is only the view of mathematical discovery that, for a number of historical reasons, has been held in the last century. From the Greeks up to at least Descartes and Newton, mathematical discovery was considered by no means illogical. Indeed, Greek mathematicians invented a method of discovery, the analytic method, and not only made their discoveries by it, but also described how they arrived at them in a number of works. As Pappus tells us, these works were called the ‘Analytic Corpus’ and were the work of three men: Euclid, Apollonius of Perga, and Aristaeus the Elder. The analytic method was also the method by which Descartes and Newton made their mathematical discoveries.

The analytic method is by no means illogical because it is based on non-deductive rules and deductive rules, so on logic. It is the fundamental method of discovery not only in mathematics but also in science. Thus, Ernst Mach says that “the greatest and most important scientific discoveries owe their origins to the analytic method” (Mach 1976, 191).

Gordon: You categorize your philosophy of mathematics as heuristic, which focuses on the notion of discovery, as the origin of the word ‘heuristics’ can be found in ancient Greek where ‘heuriskein’ is defined as ‘serving to discover.’ As readers of the JHM may well know, Pólya

considered heuristics essential for gaining mathematical understanding and defined heuristics as the ‘means and methods of problem solving.’ You extend the definition. What distinction(s) are you introducing?

Cellucci: Pólya has a restricted view of heuristics, because he claims that heuristics only “offers you good examples for imitation,” it is only “a practical art, like swimming, or skiing, or playing the piano: you can learn it only by imitation and practice” (Pólya 1981, ix). Instead, according to Pólya, the method of mathematics is the axiomatic method. Indeed, he says that, for the mathematician, the axiomatic method is “the distinctive mark of his science.” (Pólya 1954, I, vi).

On the other hand, Pólya’s answer to the question of how solutions to problems are discovered is: “The first rule of discovery is to have brains and good luck. The second rule of discovery is to sit tight and wait till you get a bright idea” (Pólya 1971, 172). This answer is incongruous because it is of the same kind as the answer of Molière’s Bachelierus: “I am asked by a learned doctor for the cause and reason why opium makes one sleep. To which I reply, Because there is in it a dormitive virtue, whose nature is to make the senses drowsy” (Molière. *The Imaginary Invalid*, Act III, Interlude III). Indeed, Pólya’s answer amounts to saying that the reason why our mind discovers solutions to problems is that there is in it a discoveritive virtue, whose nature is to make the mind inventive. Pólya’s answer is all the more incongruous as he admits that there is a method of discovery. Indeed, he says that there are procedures which are “typically useful in solving problems” and are “practiced by every sane person sufficiently interested in his problems” (Pólya 1971, 172). The best of such procedures is “the method of analysis, or method of ‘working backwards’” (ibid., 225).

In fact, an adequate answer to the question of how are solutions to mathematical problems discovered is that they are discovered by the method of analysis, or analytic method. The latter is the basis for the heuristic philosophy of mathematics put forward in my book, which is a new form of an approach to the philosophy of mathematics originating from Lakatos (1961, 1976). On the differences between the heuristic philosophy of mathematics put forward in my book and Lakatos’ see Gillies 2023, sec. 4.6.

The heuristic philosophy of mathematics provides a completely rational account of how solutions to mathematical problems are discovered. Moreover, it provides an account that, being completely rational, can be communicated, as already shown in antiquity by the works of the Analytic Corpus of Euclid, Apollonius of Perga, and Aristaeus the Elder. This sharply contrasts with the presentation of mathematics results in current textbooks based on the axiomatic method which, as Kline says, “deceives the student” by leading him “to believe that mathematics is created by geniuses who start with axioms and reason directly from the axioms to the theorems. The student feels humbled and baffled” (Kline 1970, 273–274). The presentation “omits the real activity,” it “conceals the flesh and blood,” therefore it “is often so artificial that is meaningless” (ibid., 272). As Lakatos observes, “some working mathematicians” object that communicating how solutions to mathematical problems are discovered “would require the rewriting of textbooks, and would make” them “much longer,” but “the answer to this pedestrian argument is: let us try” (Lakatos 1976, 144).

Gordon: So, heuristics is quite essential for doing mathematics, but it’s not the whole picture of how to better understand the engagement of mathematics – in particular, the method of mathematical research.

Cellucci: As appears from Pólya, one can believe in the importance of heuristics for teaching and at the same time believe that the method of mathematics is Hilbert’s axiomatic method. Therefore,

between heuristics and the method of mathematics, the primary question is the method of mathematics.

From the antiquity until the second half of the nineteenth century, mathematicians practiced the analytic method as the method of mathematics. Euclid considered the axiomatic method only a method useful for teaching the most elementary parts of mathematics to beginners. As Proclus tells us, in the *Elements*, Euclid only included “theorems and problems that are worked out for the instruction of beginners” (Proclus, *In primum Euclidis Elementorum librum commentarii*, 69.6–7).

Only with Hilbert the paradigm of mathematics changed and the method of mathematics was essentially impoverished. For, before Hilbert, the method of mathematics consisted of two parts, analysis (the heuristic part) and synthesis (the deductive part). With Hilbert, analysis (the heuristic part) was abolished and the method of mathematics was reduced to synthesis.

Gordon: Many mathematicians speak of the critical role of intuition in proving theorems by the axiomatic method. In stark contrast, you say intuition has no role in problem solving by the analytic method. Would you help us see the disparity?

Cellucci: Solving a problem by the analytic method does not involve intuition. For, on the one hand, intuition has no role in the formulation of hypotheses, since the latter are obtained from the problem, and possibly other data already available, by some non-deductive rule, so not by intuition but by logic. On the other hand, intuition has no role in the justification of hypotheses, since their plausibility is established by comparing the arguments for and against them, so not by intuition but by logic. Therefore, intuition has no role in solving a problem by the analytic method.

That solving a problem by the analytic method does not involve intuition is relevant to mathematics, because there are many examples where intuition sharply contrasts with mathematical results. Thus, intuition tells us that a ball cannot be decomposed into two disjoint balls each of which is equivalent to it. On the contrary, Banach and Tarski proved that, given a ball, there exist two disjoint balls each of which is equivalent to the given ball by finite decomposition. As Hahn says, these examples demand “the expulsion of intuition from mathematical reasoning” (Hahn 1980, 93).

Gordon: You contrast the top-down approach of the axiomatic method with the bottom-up approach of the analytic method. What should we know regarding the distinction?

Cellucci: The axiomatic method is an expression of the top-down approach to mathematics, according to which every part of mathematics must be developed from above, namely starting from axioms. The top-down approach has been the paradigm of mathematics in the past century, mainly due to the influence, on the one hand, of the Göttingen school of mathematics led by Hilbert, and, on the other hand, of the Bourbaki group. The analytic method is an expression of the bottom-up approach to mathematics, according to which every part of mathematics must be developed from below, namely starting from problems.

An example of these two approaches to mathematics is provided by Leibniz’s and Newton’s approaches to calculus. Leibniz’s approach was top-down, Newton’s approach was bottom-up.

Gordon: Every body of knowledge contains definitions of relevant terms. As you share, traditional practice in prior philosophies of mathematics contains a stipulative view of definitions. How would you distinguish your heuristic view of mathematical definition with traditional practice?

Cellucci: The view that mathematics is problem solving by the analytic method provides an approach to all main aspects of mathematics that is completely different from the approach of the view that mathematics is theorem proving by the axiomatic method. This includes mathematical objects, proofs, definitions, diagrams, and notations.

For example, as regards mathematical definitions, the view that mathematics is theorem proving by the axiomatic method holds the stipulative conception of definitions, according to which definitions are mere abbreviations. Thus, Hilbert says that a definition is “a mere explanation of signs” and is only “the introduction of an abbreviating symbol for a complex expression” (Hilbert and Bernays 1968–1970, I, 292). On the contrary, the view that mathematics is problem solving by the analytic method holds the heuristic conception of definitions, according to which definitions are hypotheses that are made to solve mathematical problems by the analytic method. Like all hypotheses in the analytic method, such hypotheses must be plausible.

The heuristic conception of definitions has several advantages over the stipulative conception. For example, the stipulative conception does not account for the fact that finding a suitable mathematical definition can make the difference in discovering a solution to a problem. For, according to it, a definition is only an abbreviation. On the contrary, the heuristic conception accounts for this fact because, according to it, a definition is a hypothesis, and finding a suitable hypothesis is a crucial step towards finding a solution to a problem.

As another example, the stipulative conception does not account for the fact that mathematicians often give definitions that afterwards turn out to be incorrect. Thus, Jordan gave a definition of a curve that corresponded to the idea that a curve is what is generated if a point runs along in continuous motion, but Peano found a counterexample to it. The stipulative conception does not account for this fact because a definition, being merely an abbreviation, cannot be incorrect. On the contrary, the heuristic conception accounts for this fact because, according to it, a definition is a hypothesis that is plausible, and a hypothesis that is plausible at one time can become implausible at a later time, when the arguments against the hypothesis become stronger than those for it.

*The interested reader can find an overview of the book *The Making of Mathematics* by Prof. Cellucci (2022) in his article *Introducing Heuristic Philosophy of Mathematics* (2023), published in the open-access journal *Annals of Mathematics and Philosophy*. The book can be found on the Springer Synthese Library website and the article on the journal website. Professor Cellucci, thank you very much for sharing your new paradigm for understanding and engaging mathematics!*

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