PROMOTING MATHEMATICS STUDENTS' AGENCY

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Introduction

While it is tautological that the future is in the hands of the young, research regarding student alienation and anger (cf. the Pew Report research, Lin et al. 2024; the Islamabad Council research, Iqbal and Zahoor, 2024; and the paper on education by UNESCO, 2019) make clear that the future functioning of society is unclear. Student mind-sets would hopefully be informed and shaped by an education that promoted their productive thinking and provided an environment within which their constructive collaboration was given support. From that vantage point, how the mathematics curriculum can support student problem-solving and social development is the focus of this paper. And deservedly so, for while student agency has been a valued and sought after goal in the mathematics classroom (cf. Ernest, 1998; D'Ambrosio, 2024; Skovsmose, 2021), it is found to be the most difficult and least appealing subject by many students. Educational research continues to locate mathematics as that subject matter which promotes stress, anxiety, and even phobia for students (cf. Burns,1975; Hembree,1990; Hopko, et al., 1998; Jackson & Leffingwell,1999; Boaler, 2008; Ernest, 2008; Ashcraft & Moore, 2009; Hersh & John-Steiner, 2011; Maloney & Beilock, 2013). Inasmuch as this problematic valuation has held rather constant over decades, it is reasonable to conclude that it is structural, for mathematics involves finding patterns and relationships and using reason to make judgements, which is what human beings are wired to do as those mental actions are essential for survival.

The problem in part can be seen to be that student interest doesn't tend to impact curricular decisions, which suggests the need to create classroom practices that give students greater voice in pursuing self-chosen interests alone and collaboratively. Not only would it be of value in establishing a more vibrant mathematics experience, but it could well have further value with students becoming citizens having experience when working on self-chosen societal concerns. Toward that socially valuable development, multiple-centers investigations will be considered. In this setting, students have opportunity to work with as well expand upon the teacher's focus question so as to create a learning environment where student greater interests promote their making dedicated problem-solving and collaborative efforts. Additionally, to support students gaining greater understanding of and capacity for doing mathematics a language of inquiry will be considered, as the formal language in mathematics textbooks and classrooms tend to be associated with the presenting of mathematics, the products of inquiry, which is other than the heuristic language associated with engaging mathematics successfully (cf. Gordon, 2024). With classroom emphasis given to investigations, students can come to see mathematics more completely, "as a human activity", as Hans Freudenthal sought, for the language of heuristics can be seen to support the doing of mathematics which is other than the language of presenting mathematics that tends to populate mathematics textbooks. With the incorporation of this dual framework of multiple-centers investigations and heuristics that underlies the doing of mathematics, mathematics classroom experience can promote the opportunity for students to develop themselves as more capable learners as well more valuable and valued participants working with others. Such an environment shifts the focus from the primacy of a priori chosen subject matter to matters regarding the quality of the lived mathematics experience in the best interests of the student, the school, and society (cf. Dewey, 1973). To clarify the value and operation of this framework it will be necessary to consider some mathematical content. As this journal is given to philosophical considerations, the content will be introduced sparingly while the discussion will hopefully elucidate the nature of mathematics classroom practice directed to promoting and supporting student agency as more capable individuals and more socially aware and responsible participants.

Multiple-centers investigations

Dewey recognized ". . . the very process of living together educates. It enlarges and enlightens experience; it stimulates and enriches imagination; it creates responsibility for accuracy and vividness of statement and thought" (1916, p. 6). That understanding makes clear why autocratic leaders and societies naturally and inevitably become oppressive. And so the necessity for promoting a vibrant collaborative engagement in the mathematics classroom for the positive personal and societal gains that could be obtained and which can be understood to be essential as regards the continuing functioning and developing of a society that supports the individual and collective aspirations of its citizenry.

Having the opportunity to sit facing each other promotes interactions where students can listen carefully to one another and reflect on questions, conjectures and claims shared, even challenges to one's own thinking. It is not unrealistic to expect in such a setting that everyone would respect the collaborative investigative effort, for what is uncovered as a consequence of the give-and-take is often what none of the parties alone would have realized as the resolution. Additionally, working in groups can provide a setting for students to ask questions and play out ideas they're not particularly sure of or comfortable with "announcing" to the whole class. As well, it provides opportunity for continuous feedback – i.e., an ongoing "ungraded assessment", from which all can learn to become more reflective participants.

Research supports collaborative practice as a productive means for student learning (cf. Harvey & Daniels, 2009; Chin & Osborn, 2008). It makes educational sense that the classroom practice would have the goal that every student has the opportunity to have their voice heard and be valued in promoting collective understanding. For grouping represents in microcosm what will extend to student future experiences when as adults they would be working together to hopefully make things better for themselves and the rest of us. With providing opportunities for group investigations, mathematics educators can promote student's social and intellectual development, and in so doing provide an experience where constructive student dialogue is prologue to a vibrant pluralistic society.

Yet, having small groups of students engaging the curriculum material simultaneously can be quite a challenging pedagogical experience (cf. Tomlinson, 2001), as grouping students is not as simple as dividing the number in the class by a single-digit divisor. In one grouping format, with students of similar academic abilities working together, there are potential gains as well losses. The interaction would most likely be quite productive for the more mathematically inclined students; but clustering students this way means there is little if any opportunity for interactions with students of varying ability, and those students who struggle would only have each other to count on. As such this can be seen as a questionable if not problematic setting for a future democratic society where people of varied abilities and backgrounds would need to work together productively. A grouping framework that does respond to the range of student abilities has the more able and/or more mathematically inclined helping those who tend to have difficulties learning/connecting to the material. While gains would likely be made by those who need more assistance, the more capable and/or more dedicated would likely not have the opportunity to engage the material as deeply or with as much commitment as they might hope due to the time given to their working with others requiring their attention. Additionally, the less able or less inclined would regularly be on the receiving end of the learning experience, and so most likely feel they haven't truly added to the group's understanding. As such, this well-intentioned practice could fail to foster productive interactions and positive feelings for participants.

When students are actively engaged in their learning, "the assumption of the externality of the object, idea, or end to be mastered to the self" (Dewey,1975/1913, p. xi) is seen as epistemologically misguided. Naturally, students want to become capable actors in shaping their lives, so it makes institutional sense that "educators should be encouraged to focus more strongly on facilitating interest . . . . [And] If a higher level of interest is desired, then instruction should involve more active and student-centered activities" (Schiefele and Csikszentmihalyi, 1995, p. 179). Multiple-centers investigations support that effort and development, as students are grouped by interest and so are connected by the desire to work together on shared concerns. As Dewey and the rest of us have recognized, interest promotes effort (1975/1913), and in this investigatory setting, student interest promotes social control and individual initiative as students collaborate with others who share their interests. In this setting, it is to be appreciated that "While what we call intelligence may be distributed in unequal amounts, it is the democratic faith that it is sufficiently general so that each individual has something to contribute, and the value of each contribution can be assessed only as it enters into the final pooled intelligence constituted by the contribution of all" (Dewey quoted in Ratner, 1939, p. 403). This awareness provides a lens, a corrective lens, in establishing grouping practices with an eye toward promoting everyone's participation, acceptance, and growth.

By recognizing student interests, multiple-centers investigations provide the opportunity for students to pursue inquiries of their choosing as the direct consequence of the teacher's focus question. In general, as often occurs in the classroom, student investigation begins with the teacher introducing a question and students working alone and in groups in an attempt to resolve the question. But, as experience makes evident, after a while students become aware of how interested they are in continuing to work on the question, or if they are actually more interested in an associated question, or an activity that came to mind as a consequence of their initial efforts. Multiple-centers investigations under the guidance of the teacher create the opportunity for students who share similar interests to form new inquiry groups that recognize their concerns and interests. In this way greater connection is made in the best interests of the student, the school and society, as will be discussed. The practice reflects the Dewey Laboratory School philosophy where there was ". . . emphasis upon the importance of the participation of the learner in the formation of the purposes which direct [their] activities in the learning process" (Mayhew and Matthews, 1963/1938, p. 67).

When the class is brought together to discuss the various findings, students get the chance to present the work they did alone and together, and every student can be appreciated for what they brought to the collective understanding. Questions are asked until an acceptable level of understanding of all the investigations, as determined by the teacher, is gained by everyone. In this way, students grow in appreciation of each other's concerns and efforts, and come to gain an understanding of how working together supports their interests and expands their knowledge. The teacher can then make point of those findings that deserve added attention, and may decide to create an examination in acknowledgement of the curricular elements that were shared.

To make more explicit how multiple-centers investigations evolve, classroom experience associated with why there are 360 degrees in a circle follows. For readers interested in one given to younger students where student investigations uncover patterns and the natural inclination to formulate algebraic relationships see Gordon (2018), and for those readers interested in one for older students which involves exploring parabolic arcs that populate park fountains where velocity and horizontal distance are parameters that need to be introduced into a quadratic equation, see Gordon (in press).

Investigating the standard textbook claim of a 360^{o} circle

Multiple-centers investigations promote and support initiative, inquiry, and collaboration - dispositions and practices essential for a prospering democratic society. What follows is a brief description of questions and responses students shared that served to establish a multiple-centers investigation as the result of the teacher asking, "Have you ever wondered why it is claimed there are 360^{o} in a circle?" Some students suggested that it was a matter of fit, until it became clear that a degree as any other unit of measure is arbitrary. That awareness promoted interest in finding how the length of a foot was determined, and more subtly a mile, and other measures the teacher and students appreciated – that is, that established a good use of student effort and classroom time. Some students wondered why it wasn't 365 units since the symbol for a circle seemed to represent a path around a center point like the Earth orbiting the Sun. A student who often found mathematics disconcerting suggested that 400 units would have more appeal as it would make for a quarter-turn of 100 units not 90 – which agreed with the 18^{th} and 19^{th} century mathematician and scientist Simon Laplace and which is presently used in civil engineering. Why the latter measure wasn't also used in the classroom became another consideration for student attention. Some students wondered what society first used a 360^{o} circle as well a 365-day solar year. And some remained interested in coming up with a rationale for there being 360 equal central divisions. In this way, student interest created investigative groups where their coming to know was the consequence of their initiative and inquiry effort one of raising interesting questions which would likely not be part of a classroom conversation where the curriculum was pre-determined. (The interested reader can find a detailed version of this investigation in Gordon 2022.) In general, such focus helps students understand that it is a good idea to ask about how mathematical claims, and by extension societal claims, came to be. Questioning the taken-for-granted can also be of value in supporting more life-enriching practices, which would seem to be associated with the gaining of an education.

Heuristics – the language of investigation

Dewey makes point of the excessive commitment of educational institutions to the presentation of subject matter as it diminishes the role of the student and society. He sees the imbalance as a "shortcoming [that] springs from a disposition to treat knowing as though it were entirely separate from doing" (Dewey 1973, p. 193). That perception can be said to be true regarding the presentation of mathematics, as formal mathematics presentations are constituted by a language of demonstration which is the result of having thought things through. In that setting, the reader is left to wonder how the argument was constructed.. The direct consequence is that students are presented with mental actions such as "setting the equation equal to zero", "eliminating a variable", "changing the form of the rational function …", etc., which represent what has been found to be of value in solving specific problems, but without including any insights into how those decisions were decided upon.

To engage problems successfully requires drawing upon general problem-clarifying strategies (heuristics) which represent a language of investigation that are essentially life-enriching practices. In the effort to solve problems, mental actions associated with the process of inquiry not the product of inquiry are drawn upon, such as *make the problem simpler*, *take things apart*, *tinker*, *change representations*, etc. From this understanding, knowing and doing are unified. But with the informal language of inquiry given little consideration if any in mathematics textbooks, students aren't informed of what is involved in the doing just the knowing.

Research supports the application of heuristics for engaging mathematics (Lucas, 1990; Bodner & Goldin, 1991; Schoenfeld, 1992; Cuoco et al.1996; Kramarski et al., 2002; Chavez, 2007; Charbonneau et al., 2009; Matsuura et al., 2013; Gordon 2011, 2021). Additionally, a recent study by Singh et al. concluded that "the results depicted a significant increase in the mathematical thinking post-test score among the students who underwent a seven-week pre-post problem solving heuristic treatment" (2018, p. 289). But, as mentioned, mathematics as traditionally presented does not include the language of investigation that serves to promote the underlying inquiry essential for doing mathematics. For example, students come to be informed of the practices and application of Proof by Cases, "isolate the radical", and the Method of Partial Fractions. Yet, what remains hidden is that all three seemingly disparate procedures are the result of applying the heuristic of *taking things apart* in an effort to solve a problem.

In general, formal presentations, including proof demonstrations, definitions, properties, and procedures, tend to be absent of the informal inquiry language that was critical to establishing their representation. And with students' mathematics experience beyond the early years often a reflection of the commitment of the mathematics community to an aesthetic of brevity, it makes sense that students would tend to have difficulty learning mathematics due to its succinct presentation framework given to demonstration. Mathematics textbooks tend to introduce, for example, the Pythagorean Theorem with the relationship of the sum of the squares of the sides equaling the square of the longest side. Often left undiscussed is how might anyone have come to that. And why would that be a focus at the time of around 600 BCE? And why a "hypotenuse" instead of just the longest side? And why a "right" angle - can't it point to the left?

For a more extended instance, students are presented with the slope of a line in the Cartesian plane defined as "the change in y over the change in x", while "the change in x over the change in y" tends not to be considered in traditional American mathematics textbooks, though it would seem a most natural consideration students could well have come upon. Were students comfortable drawing upon heuristics and, in particular of *looking/describing carefully* and *tinkering,* the teacher could begin an investigation with drawing lines in the coordinate plane, and students how they might mathematically differentiate one from another. Since they would recognize that the rate of change was constant for all lines, they could well come to hold two plausible hypotheses for the defining distinction – the "change in x over the change in y" and the "change in y over the change in x". In the discussion/investigation given to evaluating both, they can come to see the more appropriate relevance of the latter and connect their knowing with doing.

Such experience provides the opportunity to realize that "Definitions in mathematics are not starting points but arrival points in the solution to problems" (Cellucci, 2022, p. 296). A case in point, In "Manufacturing a Mathematical Group: A Study in Heuristics", 2018), Ippoliti apparently looked and describe carefully the workings of mathematicians in their effort to create a definition for a mathematical group. His dedicated investigatory effort found the heuristics of *look for similarities*, *change of representation*, *generalize from particulars*, and *reason by analogy* that Lagrange, Cauchy, Galois, and Cayley apparently drew upon as mental actions in its development. Reflecting on his findings, Ippoliti concluded that "Focusing on the heuristics that gradually have led to its formation and refinement . . . displays paradigmatic features of the core of problem-solving" (p.1). In stark contrast, the traditional textbook introduction of a mathematical group provides the definition accompanied by illustrative examples. With all definitions presented in like manner, as is the case with the slope of a line, students would seemingly have little evidence toward understanding the constructed nature of mathematics and the opportunity to be actively involved or appreciative of the effort in creating definitions.

Dewey recognized that looking was a creative act and, with *looking and describing carefully* being made explicit content, students can come to have a more complete understanding and appreciation of the formulation of the linear equation with the investigation determining the slope formulation. Additionally, it would seem reasonable to raise the question as to why Descartes would have chosen *m* to represent "slope", as that would hopefully also be a concern of interested students. Otherwise, the seeming disconnect may well promote a mathematics experience where students find themselves unable to make sense of what they are being told, which naturally tends to promote alienation. More completely, to tacitly acknowledge the value of asking questions, it would seem discussion would also include consideration of the symbolic representation associated with the linear equation, "y = ax + b", along with the replacement of *m* in place of *a*. Those elements can be seen as the result of thoughtful decision making and, left unconsidered would seem to promote only student confusion and alienation.

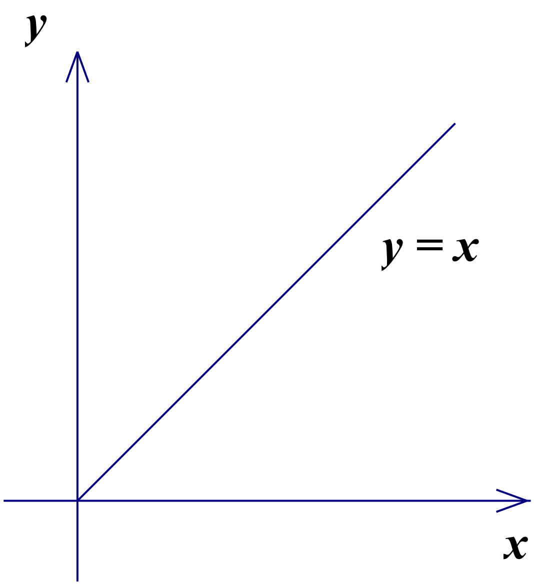
More completely, as "Mathematical ideas are discovered through an act of creation in which formal logic is not directly involved" (Hanna, 1989, p. 22), heuristic reasoning can be understood to play a foundational role in successful investigatory efforts. After all, "Mathematics . . . is never deductive in its creation. The mathematician at work makes vague guesses, visualizes broad generalizations, and jumps to unwarranted conclusions" (Halmos, 1968, p. 376). It is mental actions such as these that ultimately lead to the construction of mathematical knowledge, as will be made evident in considering the essential role of heuristics in the context of demonstrating mathematical proofs.

The absence of heuristics in the formal presentation of proofs can be seen as the "context of discovery" not accompanying the "context of justification". It has a long history (Van Bendegem, 1993) and a problematic one. The difficulty considered here is that the language of investigation that was essential in constructing the formal argument, heuristics, is other than the language of demonstration that constitutes the formal argument, so the reader seeing the final explication may well have little if any idea as how it came to be. And that bifurcation has apparently been in effect sanctioned by the mathematics community, as constituting a "good" proof is that it contains a convincing argument *that* something is the case along with an explanation *why* it is the case (Hersh, 1993; Thurston, 1994; Rav, 1999; Byers, 2007). Yet, the essential mental actions that constituted *how* the proof came to be are not part of the definition. Viewing proofs through a heuristic lens makes understanding more possible not only of the proof under consideration but provides for introducing general strategies that can be applied in other efforts at proving. As a direct consequence, the opportunity for gaining greater awareness and facility engaging mathematics are obtained when heuristics is made subject-matter content when proofs are presented.

For a quick example, applying the heuristic of a *change in representation*, an argument can be made to explain the following visual proof (Figure 2):

Figure 2

A Visual Proof that $[0, 1]$ is Equipotent to $[1, \infty)$



More completely, heuristics comes from the ancient Greek, *heuriskin*, and means "serving to discover" (cf. Gordon, 2019). Yet one need not know the origins of the definition as it has universal application. Halliday (1975) found children's language to include a "heuristic function" which he described as the "language that is used to explore, learn and discover", which is naturally called for in investigations. That is, it is natural to the thinking that we all do to inquire productively, yet it remains distinct from the formal communication of mathematics in textbooks, journals, and classrooms. And that omission is and has been a major deterrent to student understanding of and doing mathematics and, in particular, with regard to proofs.

Weber's 2003 essay, "Students' difficulties with proof" for the Mathematics Association of America, shares how problematic teaching proofs has proven to be. His presentation begins: "Proof is a notoriously difficult mathematical concept for students", even for students having taken "proof-oriented courses" in high school geometry and college in introduction to proof, real analysis, and abstract algebra, who were still “unable to construct anything beyond very trivial proofs”. Moore (1994) corroborates those observations with his study of undergraduates taking a course in transitioning to proof, as he found that “All of the students said they had relied on memorizing proofs because they had not understood what a proof is nor how to write one” (p. 269). The communication problem is experienced by professional mathematicians as well. Hanna (1989, p. 22) draws upon the supporting perspectives of Ulam, Manin, and Davis, who share how difficult it was to understand formal proofs as well a former editor of *Mathematical Reviews* who shared that it could be that half the proofs published were false.

Lakatos (1976), Hersh (2014), and Cellucci (2022) have argued for the incorporation of heuristics with proofs, and the rationale for doing so is quite evident. Heuristics can be considered as the collection of problem-clarifying general strategies that are essential in making progress when it is not clear what problem-solving procedure to choose (cf. Polya, 1962; Bell, 1993). Were the critical decisions associated with the inquiry process to accompany the product of the inquiry, the polished demonstration, It would surely help in promoting the reader’s understanding.

A formal demonstration accompanied by heuristic annotation follows. Communication ethics suggests the needed incorporation goes beyond epistemological considerations with regard to students’ mathematics experience. In support of that commitment, Cellucci (2022) has argued that mathematics presented as the result of theorem-proving is not a valuable way to think of the producing of mathematics; it is problem solving which is the driving force. Problems associated with demonstrating theorems to frame the mathematics experience in comparison to solving problems will be made apparent in the proof to be presented.

The sum of the angles proof

The triangle, being the building block of all convex polygonal figures, would seem to deserve early consideration in students' study of plane geometry. However, the traditional textbook proof that the sum of the angles of a triangle is 180^{o} appears as the result of Euclid's proof in his *Elements* (Book 1, Proposition 32), which first requires the consideration of parallel lines. As readers know, the demonstration begins with "Draw a line through a vertex parallel . . .", and the proof follows as the consequence of that mental action given legitimacy by the invention of his 5th postulate. While the proof provides a convincing argument *that* the sum of the angles is 180^{o} and an explanation *why* as a consequence of parallel lines, *how* the argument came to exist is not discussed. As Einstein remarked with regard to the proof, "something deep and mysterious is hidden here". How the first step might have been decided upon would seem to deserve attention as it is makes the proof possible and, more completely, helps students in their problem-solving efforts in thinking and doing mathematically. The opaqueness can be seen as the consequence of the results being presented as the result of theorem-proving not problem-solving. If it is revisited as an effort to solve a problem, greater awareness can be gained. Discussion of how Euclid might have come to the decision for creating the Parallel Postulate would make the demonstration more realistic and mathematically more instructive – in effect, help students see how knowledge is constructed not just demonstrated, and remove the confusion that promotes their thinking poorly of their mathematical capacity.

Some history can be drawn upon to make the experience more informative. Thales and Pythagoras had been to Egypt and knew that the sum of the angles of a triangle had been experientially determined to equal two right angles, but the demonstration remained unproven. As early abstract mathematical thought developed from discussions and demonstrations associated with diagrams (cf. De Young, 2009), it seems reasonable to imagine that the ancient Greeks in trying to determine the triangle-angle sum would have drawn upon the heuristic of *taking things apart,* and with compass and straight edge arranged replications of the triangle angles to find a straight angle. With further corroboration, the potential proof maker(s) would have used the heuristic of *generalizing from the particular* to provide the impetus to try to prove that *every* triangle angle sum was 180^{o} (which is not apparent given the various shapes the figure can take). The problem they faced was how to demonstrate that the straight angle of 180^{o} had the same sum measure as the non-linear triangle angles. Euclid's determined imaginative effort, informed it would seem by *tinkering* and *visualizing*, that ultimately resulted in the creation of the 5th postulate, would be rewarded: the triangle angles and the straight line angles in conjunction with parallel lines.

In the absence of consideration of the mental actions involved with solving a problem, the formal presentation would provide a demonstration yet not seem to provide clarifying understanding. Instead, confusion and alienation could well be logical emotions and memorization the logical solution, as the first step is presented as if it were the natural place to begin. Such a presentation in the form of theorem-proving has unfortunately left students questioning their own capacities rather than realizing there was an investigation underlying the first step that would be best shared.

As noted, the formal presentation of mathematics tends to be framed in a proof framework absent of heuristic mental actions essential to the argument's construction. In this way, mathematics is seen as an art to the exclusion of the science that it also is, and as an unfortunate consequence the thinking that it is only made possible by an exceptional few. Presenting an heuristic-annotated version of the proof understood as a problem that was solved helps students see how the argument may well have been constructed. In this way they can gain insight into the heuristic considerations associated with the active language of investigation, and come to have greater facility in the construction of mathematical knowledge.

To give students practice with working with heuristics of course makes sense as it would support their thinking as how to engage non-routine mathematics problems. With that in mind, students could revisit the sum-of-the-angles consideration but begin by considering a Rectangle Existence Postulate in place of the Parallel Postulate. As the former doesn't require the introduction and study of parallel lines and associated angle relations as the form is embedded in the everyday of common observations, it offers a more direct approach than the traditional argument. And it can be seen to highlight the heuristic application of mental actions in an effort to *make the problem simpler* (Gordon, 2023).

In sum

Research supports the observation that for many students their mathematics experience has been found wanting. A classroom environment where students have opportunity to be more active participants in shaping their mathematics learning and interacting is available. As discussed, multiple-centers investigations promote students working with other students who share their interests. That supportive environment makes possible their developing behavioral practices and emotional and attitudinal capacities that reflect effective self-direction with regard to doing mathematics and the growing capacity to work productively with others in a collaborative effort. In this setting, subject-matter concerns and democratic ideals are unified. Additionally, introducing heuristics provides a language of inquiry as being an integral part of the subject-matter content which promotes students becoming more capable engaging mathematics.

However, what mitigates against the multiple-centers consideration is the pressure to “cover the curriculum”, and against heuristic incorporation the prevailing belief of the mathematics community that “The more you have to put into an argument, in terms of prerequisite knowledge, the more elegance the argument loses” (Dreyfus and Eisenberg, 1986, p. 3). Both can be seen as a commitment to an aesthetic that prioritizes the integrity of the demonstration, not the making of mathematics and the quality of the communication. The transmission holds efficiency as the prevailing ideal which locates Dewey’s concern of the problem of knowing being separated from doing and makes evident why Cellucci has argued that mathematics be considered the result of engaging problems and not of demonstrating proofs. What needs to be appreciated is that with the incorporation of heuristics as a constant in the mathematics conversation, students will have greater facility in how to engage problems more effectively, and so the time taken in group efforts and classroom discussion will be more efficiently spent.

There is a “logic of discovery” (Cellucci, 2022) which is foundational to problem solving that is available for the “making” as distinct from the “presenting” of mathematics. Left unconsidered, it makes sense that students will continue to find mathematics the most confusing study. As long as the mathematics community considers the process of discovery and heuristic considerations extraneous to the presentation of formal demonstrations, students can be expected to continue to memorize textbook and classroom demonstrations of problem-solving procedures, definitions and proofs.

However, were students presented with opportunities to connect their interests with those of the pre-determined mathematics curriculum and work with others to gain understanding made more possible with incorporating heuristic practices, they can become more aware of what is actually involved in making mathematics and working collaboratively to solve problems. More completely, in this way the mathematics community would provide what is needed for a clarifying and ethical communication essential for solving problems and for the development of more socially responsive and successful mathematics students.

Toward that life-enriching mathematics experience, it needs to be appreciated that the distinction between "demonstrative reasoning" and "plausible reasoning" (mathematical demonstration and heuristic investigation) is not a distinction that needs to be seen as problematic. For "they don't contradict each other; on the contrary, they complete each other" (Polya, 1954, vii). And were the integrity of a proof understood in terms of a communication given to promoting understanding, mathematics as traditionally presented could be seen to be transformed. For, as presented, the "Deductivist style hides the struggle, hides the adventure. The whole story vanishes" (Lakatos, 1976, p. 151). "To use an analogy, the emphasis has been on learning to use tools and not on making furniture; and when the latter is attempted, it demands strategic capabilities - concerning planning, designing, costing, choosing materials, and selecting tools - which have not been developed" (Bell, 1993, p. 7). Hopefully, heuristics will increasingly be part of students' mathematics experience, for it can support students efforts in becoming better problem solvers, and so have a more positive experience engaging mathematics and working with others. With general problem-clarifying strategies accompanying specific problem-solving procedures so that students see the inherent analytic investigation and not just witness the exclusive axiomatic demonstration, the explanatory *how* of formal demonstrations can be made explicit and the communication made more whole.

To create an educational framework that recognizes student interest and sense making is to set in motion a mathematics experience where students are able to make personally-inspired efforts which, naturally and logically, would establish a learning environment where positive energies tend to support the learning of all. With incorporating a language of investigation In prefatory discussions to the presenting of demonstrations, students can be provided with a perspective that supports their working through the formal formulations and their being valued participants in the collective effort. For some students such practices could have a profound effect on how they see themselves as socially-valuable problem solvers and more capable human beings.

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