QUESTIONS FOR A SOCIAL PRACTICE PHILOSOPHY OF MATHEMATICS BASED ON LATER WITTGENSTEIN

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Abstract

This paper attempts two main tasks. First to elucidate the key tenets of Wittgenstein's philosophy of mathematics, and to extend them into a social constructionist and social practice based philosophy of mathematics. Second to test if this proposed philosophy of mathematics can satisfactorily answer four questions as to its adequacy. These concern first, the epistemology of mathematics and in particular the certainty of mathematical knowledge; second, the epistemic and other roles of rules and norms in mathematical practice; third, the ontology of mathematics: what are mathematical objects and how far can we accommodate their objectivity? Fourth, how far does the proposed social practice philosophy of mathematics go in satisfactorily answering the concerns of practising research mathematicians?

Methodologically, a semiotic analysis of the fundamental sign practices of mathematics is employed, distinguishing between the syntactics, semantics and pragmatics of mathematical practice. What is novel in this account is first, the identification of the foundations of epistemic certainty in purely symbolic derivations in the syntactical domain. Thus, it is acknowledged there is certainty beyond what is guaranteed by the social rules and norms alone. This is unusual in fallibilist versions of social constructionism.

Second, this approach characterises the absolute certainty of mathematical results as an epistemic value in the domain of axiology, rather than as an epistemological fact. Likewise it characterises the independent existence and objectivity of mathematical objects as ontic values also located in domain of axiology, and not as ontological or metaphysical facts. The proposed social constructionism repudiates absolute certainty and metaphysical objectivity but accommodates beliefs in them as embodying the values and goods prized by working mathematicians.

Introduction

The later works of the philosopher Ludwig Wittgenstein (1953) offers a foundation for a social constructivist or social practice philosophy of mathematics. This is '*a'* philosophy rather than '*the'* philosophy, for different versions of social constructivism are possible (Bloor 1973, Cole 2015, Ernest 1998, Hersh 1997, Kripke 1982). Further, not all social practice theories and philosophies of mathematics can be identified with social constructivism. For example, Ferreirós acknowledges the origin of mathematics in social practices but insists on objective features arising from constraints external to human agency, which he claims social constructivism often downplays. Perhaps this account might better satisfy him.

Wittgenstein's later philosophy is unsystematic, as opposed to his earlier Tractatus (Wittgenstein 1922). Thus using his later philosophy as a basis involves a great deal of creative interpretation and extension, and risks being accused of failed exegesis and distorted representation. Thus my interpretations are offered as a jumping off point for an independent philosophy of mathematics inspired by Wittgenstein's concepts and understandings, and by the implicit philosophy of mathematics in Wittgenstein's later writings. What is offered by no means accurately reflects his motley and disparate views of the issues to be addressed and questions to be answered.[1]

I have identified several key concepts, ideas, analyses and arguments in the later Wittgenstein which I propose can provide rich and fruitful conceptual foundations for a social constructionist philosophy of mathematics. These are presented below as a sequence of twelve theses on the philosophy of mathematics as inspired by the later writings of Wittgenstein and some of his recent interpreters

Thesis 1: Mathematics is constituted by language-games within forms of life.
A form of life is a social practice; which is an historical formation of persons cooperating together as they work towards shared goals. However, it must be cautioned that Wittgenstein only uses the term form of life (lebensform) and its variations a few times in his writings. Despite this, it is understood to be a key concept in Wittgenstein's social philosophy and philosophy of mathematics. "Wittgenstein offers an account of mathematics as founded in what he calls our commonly shared *forms of life*." (Wheeler 2023: p. 481). "According to

[1] Treating Wittgenstein's later writings as 'facts', any social constructivist theories (philosophies) of mathematics based on them are underdetermined by these 'facts'. Likewise, the paradox will not have escape the reader that Wittgenstein's critique of 'following a rule' applies at the metalevel in interpreting and extending Wittgenstein's philosophy to build a social constructivist philosophy of mathematics , to the extent that there can be no single correct extension that follows the implicit rules of (within) his philosophy.

Wittgenstein, mathematics is embedded in, and partly constituting, a form of life. Hence, to imagine different, alternative forms of elementary mathematics, we should have to imagine different practices, different forms of life in which they could play a role." (Schroeder 2018, p.111).

Thus, following Wittgenstein, we can say that mathematics is a human activity grounded in forms of life, which are specific embodied social practices. Language games are an intrinsic part of forms of life comprising systematic sign practices associated and embedded in any mathematical form of life. "Here the term "language-*game*" is meant to bring into prominence the fact that the *speaking* of language is part of an activity, or of a form of life." (Wittgenstein *1953*, p. 11).

"Mathematics is embedded in a world of activities language - games within forms of life; they constitute the meaning of mathematical signs." (Ernest 1998, p. 42). Because mathematics is first and foremost a set of sign practices, the idea of language games is vital to identify mathematics with social forms of life.

Thesis 2: The grammar and standards of mathematical language-games are rooted in culture and institutions.
The grammar and **s**tandards of these mathematical language games, like all language games, are rooted in the culture and institutions of the forms of life. Within the constituent practices is a network of norms and mathematical rules that arises from shared communal acceptance, from historico-cultural practices, and these are not objective facts. Thus the objectivity of mathematics arises largely from communal agreement and rule-following practices. "Grammar tells what kind of object anything is." (Wittgenstein *1953*, p. 116). "A game, a language, a rule is an institution" (Wittgenstein 1978, p 334) Thus as David Bloor (1997) elucidates, language-games are institutionally sustained: their grammar depends on social rules embedded in cultural institutions.

More specifically pertaining to mathematics, Wittgenstein asserts the following.

> "The grammar of the number-sign '2' — or, the rules for using the number 2; the two forms of expression mean exactly the same … There is no difference between asking whether numbers … might not after all really exist as spirits and asking the same thing about elves. … If 'two' is a word, why should it trouble us to call '2' a word? Numbers are number-words, or mathematical signs, i.e., that is the grammatical family they belong to, their part of speech." (Wittgenstein 1974, pp. 223–224)

Wittgenstein is treating numbers like words - i.e., their meaning derives from grammar and rules of use, not from whether they denote independent, abstract entities.

Severin Schroeder highlights Wittgenstein's view that: "Arithmetic gives us new forms of description … For arithmetic to equate two expressions is, one might say, a grammatical trick." (Schroeder 2023, p. 5) . This aligns with Wittgenstein: mathematical meaning arises through grammatical conventions within calculi, not through correspondence to a Platonic realm.

Thesis 3: Mathematical statements are not descriptions - they are standards.
The assertion that mathematical statements are normative and not declarative propositions, nor are they sentences that can be asserted to be true or false, is a radical departure from traditional epistemology and the philosophy of mathematics. It goes beyond merely saying that mathematical sentences might be contingent rather than objectively necessary. It jumps the 'is' vs. 'ought' divide so rigidly enforced since the days of David Hume (1739–40). It radically undercuts the claim that mathematics offers knowledge. Applied mathematics at best is empirical, like any scientific theory. Pure mathematics is a set of concepts and techniques, a calculus, not a body of truths.

Wittgenstein *(1953) argues that m*athematical propositions do not describe; they prescribe or enact rules. Earlier (Wittgenstein *1922) he argued that a*rithmetic is not hypothetical: it says what must be so, not what is so. And this view is argued even more strongly from his later view of grammar as a rule-governed practice (Wittgenstein *1953). In his analysis of* Wittgenstein's philosophy of mathematics *Rodych concurs.* "Mathematical statements function as standards of correctness within a practice, not as descriptions of the world." And "Mathematical theorems function as normative standards, not as descriptive truths about the world." (Rodych 2018, **§3.2**). In his detailed explication of Wittgenstein, Wagner (2020: p. 3) claims that "there are several different ways to use something as a standard. Mathematical standards include them all."

Thesis 4: Rules arise from a network of norms - mathematical rules rest on communal acceptance, and are not objective facts.
The language games that make up mathematical practice depend greatly if not wholly on rules, both explicit and implicit, determining, at least in part, what is acceptable language and sign use. The rules of mathematics rest on communal acceptance, and are not objective facts nor externally imposed necessities. The rules, both explicit and implicit, arise from and are embodied in a network of norms.

> "Mathematics forms a network of norms … by accepting a proposition as self-evident, we release it from all responsibility in face of experience." (Wittgenstein 1978, p. 211)

This normative nature of mathematics is paramount. "Wittgenstein's math is normative: rule-governed and embedded in shared social practices." (Reckwitz 2003, p. 92). A consequence is that in the much discussed topic in the Wittgenstein literature of "following a rule", the outcome of following a mathematical rule, especially an implicit rule may not be uniquely and unambiguously determined (Kripke 1982).

Thesis 5: Meaning is defined by use, and symbols are "alive in use", especially mathematical signs in everyday use.
Mathematical meanings are tied to actual usage in life, within language games in forms of life, which extend beyond mathematical forms of life. "For a large class of cases… meaning is use." (Wittgenstein 1953, p. 20). "Every sign by itself seems dead… In use it is alive." (Wittgenstein 1989, p. 44). Andrew Pickering echoes this point explicitly. "Signs only gain life through use in community practice; isolated symbols are dead." (Pickering, 1995, p. 44)

Mathematical terms are sometimes used differently outside mathematics and inside mathematics. The two uses may constrain or motivate each other, but do not determine each other. A mathematical term cannot be reduced, therefore, for this and other reasons, to a mathematical reference (Wagner 2020).

Wittgenstein clearly situates the meaningfulness of mathematical signs in their use in everyday life and practice, outside mathematics, in contrast to isolated mathematical formalism.

> "It is essential to mathematics that its signs are also employed in mufti. It is the use outside mathematics, and so the meaning of the signs, that makes the sign-game into mathematics." (Wittgenstein 1978, p. 257)

Thus Wittgenstein's view is that mathematical signs and propositions gain meaning not through purely formal mathematical functions, in official uniform, so to speak, where they operate as grammatical rules within language games. It is in their everyday uses and contexts, where the signs are clad in civilian clothes, that is 'in mufti', that they are meaningful.

This is strongly echoed by authoritative secondary sources, including Juliet Floyd, Michael Dummett and Crispin Wright.

> "For Wittgenstein, the application of mathematics in ordinary life is what gives it meaning, whereas purely formal mathematical systems, detached from use, risk becoming mere 'calculating games.'" (Floyd 2001, p 290).
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> "Wittgenstein insists that the meaning of mathematical expressions is not fixed by their place in an abstract system, but by their role in human activities - measuring, counting, buying, selling - where they are indispensable." (Dummett 1978, p. 167).
> "Wittgenstein's later philosophy of mathematics stresses that mathematical propositions only have meaning insofar as they are embedded in a form of life. The symbols of pure mathematics, divorced from application, are like 'wheels that turn without moving anything.'" (Wright 1980, p. 152).

These scholars emphasize Wittgenstein's distinction between: *Meaningful mathematics* (embedded in everyday practices, measurement, trade, etc.), and *Formalist mathematics* (seen as a self-contained game, lacking meaning without application).

The key point is that Wittgenstein prioritizes *use* over *formal derivation* in assessing mathematical meaning, and that any credible philosophy of mathematics based on Wittgenstein must acknowledge everyday applications and use as the primary sources of meaning in mathematics.

Thesis 6: Mathematics is fundamentally social - no private language exists. Rule-following and proof are public, not private acts.
Mathematics is not and never can be a private construction in somebody's mind, contra Brouwer (1913), and even contra Lakoff and Núñez (2000) who try to found mathematics on private embodied schemata. Wittgenstein (1953) argues that if a sign can be given a meaning only in virtue of a language, then no private object and no private language is possible. Saul Kripke explicitly applies this conclusion to mathematical rules, too. "Rule-following is sustained by communal agreement; private interpretation cannot determine rule application." (Kripke 1982, p. 33).

As Wittgenstein says: "To obey a rule, to make a report, to give an order, to play a game of chess, are *customs* (uses, institutions)." (Wittgenstein 1953, p. 81). This not only shows that in his view rule-following is public, but that following a rule is not necessary in any metaphysical or absolute sense.

> "No course of action could be determined by a rule, because every course of action can be made out to accord with the rule … What this shews is that there is a way of grasping a rule which is *not* an *interpretation*, but

> which is exhibited in what we call "obeying the rule" and "going against it" in actual cases. … And hence also 'obeying a rule' is a practice. And to think one is obeying a rule is not to obey a rule. Hence it is not possible to obey a rule ‘privately’: otherwise thinking one was obeying a rule would be the same thing as obeying it.” (Wittgenstein 1953, p. 81)

This passage emphasises Wittgenstein’s point that rule-following is a social **practice** and cannot be grounded simply in private interpretation or individual intention. “An interpretation by itself cannot determine meaning - rule-following depends on communal agreement.” (Kripke 1982, p. 33). Likewise, rules and diagrams do not determine applications or standards. No rule, diagram or conception determines its own application or the ensuing mathematical standards, but the former do constrain the latter (Wagner 2020: p. 7)

Bloor notes that rule-followers “must, as it were, have the rule before their mind as their guide and goal” (Bloor 1997: 44). But this is not an introspective mental state: it is a shared institutional stance, learned and reinforced through communal practice.

> “Treating conventions as unverbalised habits is a better and deeper idea. This follows from the fact that the use of a rule is itself conventional, but cannot (in the last analysis) be governed by verbally formulated rules. … He [Wittgenstein] always stresses the consensus of action rather than the consensus of opinion or belief as the basis of knowledge.” (Bloor 1997, pp. 119–120)

Thesis 7: Proofs depend on derivations, but are not reducible to them. Proofs are concept-evolving acts.
A derivation pattern (that is, a formal calculation or argumentation) is not the same thing as a proof, although the former very often leads to the latter. Proofs are new standards that coordinate available techniques and standards. To prove is not the same as to establish truth. Proofs introduce something new; they do not simply capture something that was already there (Wagner 2020).

Mathematical proving and proof are part of the shared process whereby mathematical concepts and sign-usages are extended and developed. “The proof creates a new concept… by giving the proposition… a new place.” (Wittgenstein 1978, p. 173). Bangu (2023) elaborates Wittgenstein’s view that proof does not merely reveal; it shapes conceptual structure by introducing new forms, thus reshaping our conceptual framework.

> The idea that proof creates a new concept might also be roughly put as follows: a proof is not its foundations plus the rules of inference, but a

> *new* building--although it is an example of such and such a style. A proof is a *new* paradigm. (Wittgenstein 1978: p.172)

Thus a derivation becomes a proof when it is used to set new standards. Proving, therefore, is less about discovering, and more about doing something new.

Thesis 8: The disciplined techniques of mathematics are learned through communal training in mathematical practices.

The language games of mathematics practice are learned through an extended period of immersion in social mathematical activities. “Following a rule is … a practice, not something that occurs in the mind.” (Wittgenstein 1953, p. 78).

Bloor makes this more explicit. “Mathematics is performed in communities where rule-following becomes habitual through social training.” (Bloor 1997, p. 48). Ernest elaborates: “The extended apprenticeship working mathematical tasks [and problems], with expert guidance and feedback, represents an enculturation into the social practice of mathematics.” (Ernest 2023, p. 9).

The skills, knowledge and techniques of ‘doing mathematics’ correctly (participating fully in mathematical practices, which comprises the mathematical language games within mathematical forms of life) are learned through (a) preparation for such practices (learning mathematics in school and college) and (b) participation in mathematical practices, usually beginning as an apprentice (a junior or tyro mathematician in mathematical practices. Postgraduate and doctoral work in mathematics are often transitional work from stage (a) leading into stage (b), or may already represent participation in a (research) mathematical practice.

Thesis 9: mathematical ideas and objects at best share family resemblance: there no fixed essences in mathematics since meanings and uses grow and change.
In looking at the uses of a single word Wittgenstein asks if there is something common to them all. He concludes that “you will not see something that is common to all, but similarities, relationships, and a whole series of them at that.” (Wittgenstein 1953: p. 31).

> I can think of no better expression to characterize these similarities than "family resemblances"; for the various resemblances between members of a family: build, features, colour of eyes, gait, temperament, etc. etc. overlap and criss-cross in the same way.--And I shall say: 'games' form a family.
> And for instance the kinds of number form a family in the same way. Why do we call something a "number”? Well, perhaps because it has a--

> direct--relationship with several things that have hitherto been called number; and this can be said to give it an indirect relationship to other things we call the same name. And we extend our concept of number as in spinning a thread we twist fibre on fibre. And the strength of the thread does not reside in the fact that some one fibre runs through its whole length, but in the overlapping of many fibres. (Wittgenstein 1953: p. 32)

Across different language games the same words can have different meanings. These meanings also develop historically and thus change within a single practice. "We travel with the word's uses through a complicated network of similarities, overlapping and criss-crossing." (Wittgenstein 1953, p. 35). "Mathematical ideas share overlapping traits rather than rigid definitions - the family-resemblance model." (Hersh, 1997, p. 29)

If like Wittgenstein one considers the concept of number, or even the concept of number 1 (one), this has different meanings across a whole host of language games. The word 'number' can refer to a particular number, the range of everyday counting numbers, the Natural Numbers, Integers, Rational Numbers, Algebraic Numbers, Transcendental Numbers, Real Numbers, Complex Numbers and so on. In each of these uses and mathematical number structures there is a number corresponding to 1, but it is not the same number 1, for it has different uses, properties and is part of different inferences (Brandom 2000). For example, 1 is the multiplicative identity in the rationals (Q). For every (non-zero) p in Q there is a q such that $p*q=1$. But in the integers there are no multiplicative inverses so 1 cannot play this role.

Informally we act as if the sets of numbers listed above form a chain of subset (or rather substructure) inclusions, such that each number system is a substructure of the following set (except in the case of the disjoint Algebraic and Transcendental Numbers). But formally mathematicians argue that the Natural Numbers are not identical with the corresponding subset/substructure of the Integers, but is its isomorph, that is it has the same structure and preserves its properties, but also extends it, in this case. The integer $^{+}1$ corresponds to the natural number 1 but is not identical with it, although where it does not matter, we treat them as the same. Thus the same mathematical terms of signs have different meanings in different number systems and language games.

Thesis 10: Mathematical statements need not be understood as referring to mathematical objects or a mathematical reality.
To paraphrase Wittgenstein, there's little use to saying that mathematical objects pre-exist in some shadow mathematical world. To say of mathematical entities, rules or statement that they are "real", "true", "possible" (or "impossible") is not to say that they are "out there" (or "missing"), but that they are useable (or

useless) as standards. To say that a statement is "about something" is usually to say that it provides information about this thing. But statements which are said to be "about numbers" (for example) do not quite give us information about numbers. They set standards as to whether something should be called "number". (Wagner 2020). Wittgenstein rejects Platonism: mathematical statements function in calculi without referring to abstract objects. "[L]ater Wittgenstein maintains that mathematics is essentially syntactical and non-referential, which, in and of itself, makes Wittgenstein's philosophy of mathematics anti-Platonist" (Rodych 2018, §3.1.1)

> "Is it already mathematical alchemy, that mathematical propositions are regarded as statements about mathematical objects,--and mathematics as the exploration of these objects?" (Wittgenstein 1978: p. 274).

Just as we question whether alchemy can turn lead into gold, so too as we question whether mathematical alchemy can turn propositions (whether descriptions or recipes) into mathematical objects. If it can it will not be automatic, but as a result of particular transformational processes that do not refer to pre-existing mathematical objects.[2]

Thesis 11: Logic and formalisms are techniques, not foundations.
From his earliest writings onwards (Wittgenstein 1922), Wittgenstein asserts that mathematics is a method of logic, not part of logic itself, thus rejecting and contradicting the claims of Russell's Logicism. This is maintained and confirmed in his later writings. "Russell's calculus is one calculus among others. It is a bit of mathematics." (Wittgenstein 1989 p. 13). Also: "Mathematics is a motley of techniques and proofs. … The meaning of mathematical expressions is given by the rules of grammar in which they are used." (Wittgenstein 1953, p. 40). Thus mathematics is described as a collection of techniques, including logical methods, and not founded upon logic but shaped by grammatical conventions in use.

"For Wittgenstein, mathematics is a human activity—a calculus of technique—so that logic and formal systems are tools within the practice, not independent foundations." (Schroeder 2018, p. 112). Schroeder frames Wittgenstein's later view as positioning mathematics as a pragmatic toolkit, where logic is employed only insofar as it is helpful—not as the ontological basis.

"He [Wittgenstein] rejected the idea that mathematics is founded on logic; rather, logic plays a role as one technique among others in the mathematician's

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[2] This is treated in the following, where the ontological question is discussed

toolbox.” (Rodych 2018, §2.1). Thus logic is instrumental, a tool, not foundational, within mathematics.
Commentators like Rodych concur that logic in mathematics is a tool, not a foundation. “Logic is instrumental, not foundational, within mathematics.” (Rodych, 2018, §2.1)
“The mathematical problems of what is called foundations are no more the foundation of mathematics for us than the painted rock is the support of a painted tower.” (Wittgenstein 1978, p. 378).

One consequence of mathematics being founded on practice, and not on mathematical logic is that mathematics can live with contradictions.

> “Does this mean that with contradictions one must always get into trouble? As a matter of fact, it doesn’t. No one has ever yet got into trouble from a contradiction in a system that has been used carefully.” (Wittgenstein 1978 p. 362–363)

This clearly indicates that Wittgenstein sees a contradiction not as necessarily destructive but any harm caused is contingent on misuse. Indeed Wittgenstein (1978) suggests that if a contradiction were now actually found in arithmetic it would only prove that an arithmetic with such a contradiction in it could still render very good service. For historically it has proved very useful. He goes on to say “A contradiction in the rules of a game doesn't show you what you have to do (what move you have to make, for instance), but rather that the rules do not determine a move.” (Wittgenstein 1978, p. 57.

Later commentators on Wittgenstein’s philosophy of mathematics concur. “There are good reasons to try to avoid contradictions in such things as commands and descriptions, but avoiding contradictions is not always strictly necessary.” (Wagner 2020, p. 26) “Contradictions do not point to factual error but to grammatical features of calculi.” (Rodych 2018, §3.3) . “In his later writings… he held that contradictions are not false, but rather signs of misapplication of language within calculi.” (Goldstein 1986, p. 45)

Thesis 12: No text alone should set standards; practice does.
According to Wittgenstein, practice always precedes any codification in rules or text, and must serve as the touchstone for standards. In this respect there is an analogy with the linguistics of Saussure (1959), who argues that speech (parole) always precedes grammar (langue). The latter is a static codification of patterns observed or induced (not deduced) from the former.

> "Philosophy must not interfere with the actual use of language; it leaves mathematics as it is, and no philosophical text can make mathematical practice obsolete." (Wittgenstein 1953, p. 57)

McGinn (2004, p. 98) affirms that a**ccording to Wittgenstein** "Standards of meaning are established in practice, not derived from texts." Lastly, according to Roy Wagner's elucidation of Wittgenstein's later philosophy of mathematics texts "should not be used to set a system of standards." (Wagner (2020 p. 29)

Commentary
In my view these Wittgensteinian theses or proposals together represent the basis of a fully social philosophy of mathematics, far ahead of the understanding of traditional philosophers in the post World War 2 period, and ahead of what was generally acceptable at the time of his writing. Wittgenstein's philosophy of mathematics was at first treated respectfully, but as an oddity. For example the influential anthology of readings by Benacerraf and Putnam (1964) devoted a whole section on Wittgenstein's philosophy of mathematics, representing more than 20% of the content of the book, with 60 pages of selections of Wittgenstein's work, followed by another 48 pages of discussion of Wittgenstein's philosophy of mathematics in the form of three chapters.

However, the authors of these chapters fail to grasp his main point that language and mathematics are through and through social activities. These authors are still in the grip of the dominant ideology of individualism, according to which thinking is based in the minds of individuals. From this position they overlook the fact that abstract thinking is only made possible through that most social of human inventions, language. Thought has achieved its modern form, is formatted, carried and expressed through language (Vygotsky 1978). And it is only through language that the symbolic activities of mathematics are made possible.

However, less than twenty years later the second edition of this book (Benacerraf and Putnam 1983) had removed the Wittgenstein section including Wittgenstein's writings and the commentaries on them. The new edition has only ten references to Wittgenstein in the whole text. Most refer to the Tractatus, and related early ideas, and only three of these references concern his later philosophy. Thus Wittgenstein has been virtually erased from the standard philosophy of mathematics, or at least the standard bearer of 'approved' readings.

One reason for the drop in interest may be due to dominance of one of the two main interpretations of Wittgenstein's philosophy of mathematics, and indeed of his later philosophy overall. This is the individualistic interpretation, which at the time was much more popular than the social/collectivist interpretations.

> "There are also two competing schools of thought about the character of Wittgenstein's later philosophy and, in particular, his account of rule following. Adherents to one of these schools read him as an individualist; adherents to the other, as a collectivist." (Bloor 1997, p. ix).

The great weight of modern academic philosophy has come down on the individualist side. From this perspective, considerations of knowledge fall into two categories: personal subjective 'knowledge' or rather belief, and (proper) objective knowledge. From this perspective social 'knowledge' is reducible to the subjective or objective. It is either one person's beliefs or full objective knowledge.

From an individualistic perspective, some of Wittgenstein claims about knowledge and mathematics appear to be naïve, simplistic or simply incorrect. But such condemnation misses the point, according to a collectivist and social interpretation. Wittgenstein was trying to establish a radical new collectivist philosophy. From this perspective it is strikingly original and important.

The dominance of individualism in Anglo-American philosophy, and the consequent critique and rejection of Wittgenstein's philosophy as viewed from this perspective may explain why interest in his philosophy of mathematics waned in the 1960s and 1970s. From an individualist perspective Wittgenstein's philosophy appears trite and at times incorrect. However the development of more social philosophies in the works of Rorty (1979), McDowell (1994), Brandom (2000), MacIntyre (1984), Quine (1960) and even Putnam (1981), as well as others may explain renewed interest in his social philosophy. Likewise, since the emergence of Lakatos' (1976) dialectical and historical philosophy of mathematics, the 'maverick' school in philosophy of mathematics (Kitcher 1984), and even more recently social constructivism (Ernest 1998, Cole 2015, Hersh 1997) and the philosophy of mathematical practice (Mancosu 2008) have brought the social in mathematics to the fore, as a topic of interest, even if only for a minority of scholars.

What can be said is that the former neglect of Wittgenstein's philosophy and philosophy of mathematics has been rectified by more recent work. Looking at the uses of Wittgenstein's publications, according to Google Scholar in July 2025, Wittgenstein (1953) has been cited in 74,380 publications, and a Google web search reveals 46,600 references to Wittgenstein's philosophy of mathematics. Although the explicitly mathematical Wittgenstein (1978) is cited less, in 3343 publications, this is still an indication of great interest from scholars.

Thus interest in a social philosophy of mathematics drawing on Wittgenstein has grown. But the old question in a new guise rears up its head: does such a philosophy hold up?

Questions for a Social Practice Philosophy of Mathematics

Building a social practice philosophy of mathematics on the foundations of Later Wittgenstein opens itself up to some deep philosophical questions about its adequacy and merit. These include the traditional epistemological and ontological questions. First of all, there is the question of knowledge and certainty.

Q1: The epistemological question.
How does a social practice philosophy of mathematics provide a foundation for the necessity, indispensability and universality of mathematical knowledge, and guarantee that it is seen as and has the status of being, or seeming to be (to not prejudge the answer) inevitable, incontestable and unquestionable? How does it avoid making its account of the basis of knowledge one of contingency rather than necessity?

To answer this question it is useful to drawn some distinctions taken from semiotic theory. These go beyond what Wittgenstein offers in his writings, but they help to accommodate some of his key conceptualisations.

The semiotics of mathematics

Above all else mathematics is a symbolic practice. It may involve actions on the world at some levels, but these are mediated by a conceptual framework, comprising a universe of signs and sign-based activities. The language games at the heart of mathematical forms of life concern activities with signs, and their uses within mathematics, but also uses in the wider world. That is, within a range of forms of life and social activities beyond purely mathematical ones. What links all of these areas is a range of different uses of signs and symbols.

Semiotics is the study of sign systems and uses, and its concepts and analyses offer a new way of understanding mathematical language. In turning to semiotics, I draw primarily on Morris' (1945) distinction between three realms or categories of sign use, the syntactic, semantic and pragmatic domains. Syntactics focuses on signs in themselves, namely the grammar and formal rules of formation and operation of signs, without any concerns about meanings or interpretations. Indeed the syntactical realm specifically excludes all meanings and interpretations, let alone human actions, and by definition focuses only on syntax, bare signs and rules of sign formation and use as simple strings of signs.

The semantic aspect of signs concerns their meanings and interpretations including the attribution of numerical and truth values. It includes models, which are systems of interpretation for theories presented as mathematical structures, as well as applications, which use empirical interpretations of signs.

The pragmatic dimension encompasses the social, personal and human uses of signs as well as all human sign-based activities. This encompasses all the social aspects of mathematics, the beliefs, knowledge, values, and practices of mathematicians as they work with mathematical signs. It includes statements and discussions of signs, symbols, grammars, norms, rules of use, and so on. It also encompasses the use of mathematical signs throughout society whether in everyday speech, schooling, commerce, science, information and communication technologies, and so on.

Each of these three domains and its significance for mathematics and the philosophy of mathematics is discussed separately.

Syntactics
The syntactics of mathematics is the domain of purely formal calculation, derivation, symbolic string formation, symbolic manipulations and transformations. This is the domain of uninterpreted signs, for which I shall use the term symbols, and the rules for their formation and rule-based manipulation and replacement. It is a domain of univocity and disambiguity comprising just signs and mechanical operations on them. The univocity and disambiguity come from the fact that rule-based operations on signs, such as making string ‘A’ into string ‘B’, for example, are based on rules, instructions, acting as imperatives. An instruction for making string ‘A’ into string ‘B’, where the nature and structure of these strings is specified, either works or it does not. Whether there is a successful operation in the transformation or there is not is not the business of syntactics. In the later case the rule or its application are incorrect. All of the problems that of humans or mathematicians might have in following a rule, the subject of scores of books by Wittgenstein scholars, are circumvented. For the rule works, with each application enacting the required transformation. There is no ambiguity in “replace term ‘S(0)’ by ‘1’”, for example. Such rule governed behaviour represents the certainty and necessity at the heart of symbolic mathematics, for it is a language game comprising delimited finite signs and unambiguous rules for their manipulation.

The whole domain of digital computing is made possible by the syntactics of formal mathematics. No human or other decision making is involved in the running of a computer programme, once it is switched on. The enaction of any rules and sequences of instructions requires no meanings or consciousness. It is purely 'mechanical' (in fact electronic, but conducted by machine). Given any string of signs, applying a sequence of specific instructions to it results in a particular fixed symbolic outcome with certainty. The result is unambiguous, achieved with certainty, and is repeatable uniformly and invariantly, ad infinitum.

I need not add that the rules need to be applied correctly, for there is no interpretation or verification necessary. Applying the rules must automatically be correct or the rules have not been applied. There may be flaws in the sequence of instructions if they are ill-formed or do not provide the correct end string. But their operation is correct. It is just that you did not provide the correct sequence of instructions to reach *your* desired endpoint. But your intentions and your desires do not enter into the sytactical realm. You write a sequence of instruction, a programme, and it is enacted. There is nothing in this domain except signs and operations on signs.

The syntactical dimension of mathematics is what gives mathematics its certainty. The outcomes of the syntactical language games within the forms of life of mathematics are inevitable, incontestable and unquestionable. Given the fixed starting input, the application of the formal symbolic rules delivers a fixed and predictable outcome with certainty. There is no ambiguity, no doubt here, only certainty, necessity and objectivity. Here, then, is the beating heart of the certainty mathematicians and philosophers yearn for, assert the existence of, and attribute to mathematics. Social constructionism fully accepts the certainties and objectivities of mathematical derivations in the syntactic realm.

Mathematics has long been seen as both the bastion and the paradigm of certainty. This is because certainty and necessity all reside most firmly within the syntactical domain. A calculator (mechanical or electronic) and computer, or even a purely symbolic calculation or derivation with pencil and paper (or stylus and clay tablet), performed by hand, provides the required symbolic output or final state (sign), unambiguously and beyond doubt, provided only that the fixed rules for signs are applied.

As Vico (1710) wrote "The rule and criterion of truth is to have made it...We can know nothing that we have not made." His point is that certainty can only be found in humanly made systems. Although he refers to truth, which belongs to the domain of semantics, in this context we can read this as a correct derivation. Syntactical systems provide derivations that are correct with certainty.

In case this looks like sleight of hand, substituting truth for correct derivation, this happens throughout mathematics and logic. We interpret correct derivation or formal proof as the proxy for truth at the syntactic level, where truth or value (e.g. numerical value) are not available. For we only have uninterpreted signs available at the syntactical level. We also use provability at the semantic level, the interpreted correlate of derivation, as a proxy for truth.

Semantics

The semantic realm is the realm of meaning, interpretation, and values, in the narrow sense of numerical, truth and other assigned mathematical values. Within the semantic realm mathematical processes are at work on meaningful and interpreted signs. For example, a calculation is a sequence of interpreted signs each one derived from the previous one so that the sequence preserves numerical value along its whole length. A proof is a similar sequence that preserves truth values from start to finish.[3] Thus if the starting assumptions are true, this truth value is transmitted down the derivation so that the conclusion is also true. Every rule of inference in mathematics and logic is designed to preserve truth in this way. This is the basis of mathematical proof.[4]

Problem solving and mathematical modelling are also similar but they utilise multimodal sequences in their transformations.[5] Thus it is structure, that is algebraic, geometric or other forms of structure that is preserved. Structure is preserved, through changes of mode or representation in the solutions or applications sequence.[6]

The semantic realm is where most mathematical proofs are located. These are logical and mathematical arguments whereby a selection of axioms or assumed results are taken as a basis, from which, via stages of argument, a final result, the theorem, is deduced. This is effected through the employment of rules of

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[3]This can be expanded to include inequalities as well as identities, and logical equivalences as well as deductions.

[4] It is important to recognise that this property at the semantic level is what inspires and leads to the construction of rules at the syntactic level. Rules of derivation at the syntactic level are inspired by and codify the rules of calculation, deduction and proof at the semantic level. By representing and capturing such rules completely at the syntactical level there is no need for meanings and interpretations. They can be used in a purely formal way, thus eliminating all possibility of ambiguity and therefore achieving certainty. Of course this 'replacement' procedure raises the philosophical question, do the formal rules fully capture and represent the interpreted and meaning-based rules from the semantic level validly? This is a question that can only be addressed by mathematicians and philosophers in the pragmatic domain.

[5] In multimodal sequences, a verbal problem may be translated into a diagram, table or algebraic expression expressing some of its structure as part of the solution process. In applied modeling a real world or other situation is represented by a model (an algebraic theory, statistical theory, geometric model, computer mode etc) and then numerical values or parameters are fed in and predicted values of variables are read out.

[6] Mathematics has a range of structure preserving transformations including homeomorphisms, homomorphisms, isomorphisms, morphisms, geometric equivalence, symmetric transformations, continuous functions and so on. It has been claimed that mathematics is the science of invariance and structure (Klein 1893).

logic and mathematical reasoning in making (and justifying) each of the steps in the proof sequence. But such an argument must be accepted by mathematicians (within the pragmatic domain) for the theorem to be adopted as an addition to mathematical knowledge (Thesis 7).

What is accepted as a correct proof is regarded as an abbreviated argument that could, in principle, be converted into a purely formal derivation within the syntactic domain, whereby the certainty of the derivation would be established. Such a belief is of course something that belongs to the pragmatic domain. Nevertheless it serves to sustain the conclusion that any proved theorem is known with certainty.

The assumption that a logical or mathematical proof can be converted into a formal derivation depends on two subsidiary assumptions. The first assumption is that mathematical concepts can be represented fully in formal notation using the signs of symbolic logic. That this is possible was successfully demonstrated within several axiomatisations such as those of Peano, Zermelo, within the logicist programme (Frege, Russell and Whitehead), and most completely by Wolfram (2002).

Wolfram lists all axiom systems of present-day mathematics in explicit computational (syntactical) form, including logic, Euclidean geometry, Tarski's arithmetic, group theory, etc. And he says:

> "It is from these axiom systems—and particularly from the ones for logic and set theory—that most of the millions of theorems in the literature of mathematics have ultimately been derived." (Wolfram 2002: P. 773).

The second assumption is that all mathematical inferences utilized in mathematical proofs can be reduced to sequences of the basic inference steps of formal mathematical logic. This second assumption is programmatic, for it asserts first, that all existing mathematical proofs be rendered formally using only the basic logical proofs steps as derivations in the syntactical domain, ensuring the soundness of the proof. It also asserts that all future proofs can also be so rendered. Thus while this second assumptions is unprovable and ambitious, it is nevertheless, in principle, not implausible.[7]

On the basis of these assumptions, it can be asserted that the theorems and results of mathematics can be represented formally, in the syntactical domain,

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[7] A further assumption is that the formal derivations of mathematical theorems can be checked for correctness. Most such derivations would be very long. Although it might be claimed that computer software could do the laborious checking task, we would then have the further task of checking that the software itself always functions correctly.

and the derivations that represent their proofs are correct. With these provisions the theorems are known with certainty. This does not establish that they are true. What it establishes is that on the basis of the adopted bases of the derivations (‘axiom’ and ‘theorem’ strings) the conclusion as formally derived follows correctly. Interpreted in the semantic domain this means that given these assumptions (specific axioms, etc) there is a formal proof of the conclusion (specific theorem). This adds further to the strength of the widespread conviction that mathematical knowledge is objective and known with certainty, even though it contains the ineliminable element of faith that all future proofs can be rendered fully formally.

Pragmatics

The pragmatic domain is that of the social use of mathematics, both within mathematical research language games, and outside them in everyday forms of life. Mathematical practice in its fullest sense encompasses the pragmatic domain with all the social uses and discussions of mathematics and its texts, all the mathematics-related behaviours of mathematicians as well as learners, teachers and users of mathematics and its symbols and language. It also encompasses the more restricted semantic and syntactical domains, which are the locus and focus of the mathematicians’ work, employing the symbolism of mathematics itself, as well as metalanguages for discussing all of these elements.

This is also the domain that imposes restrictions and constraints on mathematics, beyond the internal rules. For the formal mathematical theories employed in the semantic and syntactic domains, especially in less advanced cases, correspond, at least in part, to everyday mathematical uses, practices and mathematical language games as employed and practised in everyday and extra-mathematical forms of life. Take number as an elementary example. The words ‘one’, ‘two’, ‘three’, and so on in various linguistic formulations have been in use by humanity for a long period of time, very likely scores of thousands of years. So the elementary relations and structures of verbal number and its usage provide a constraint on the mathematical development of number, calculation and number theory. Thus, for example, any valid mathematisations and formalisations of number must respect the total ordering of number (one precedes two, two precedes three, and so forth) and compositional facts and structures (one plus two makes three), and so on. Similar constraints operate in geometry, measurement and other elementary areas.

In addition to the constraints imposed by extra-mathematical language games and forms of life, there are many complex dimensions of mathematical pragmatics, since it contains all the human dimensions of mathematical practices. Philip Kitcher draws on Kuhn’s (1970) discussion of scientific practices in observing how exemplary proofs serve as paradigms (in the narrow sense) for scientific problem solving. At the heart of mathematical practices (comprising language games embedded in mathematical forms of life) are activities with signs and knowledge representations. These employ the practical knowledge and values of mathematicians that are acquired through extended immersion, participation and apprenticeship in such practices (Wittgenstein Thesis 8) Kitcher proposes that the practical knowledge of mathematicians within mathematical practices include

> "a language, a set of accepted statements, a set of accepted reasonings, a set of questions selected as important, and a set of meta-mathematical views (including standards for proof and definition and claims about the scope and structure of mathematics)." (Kitcher 1984: p. 163).

Thus Kitcher argues that the mathematicians' skills include the following, which have been expanded in Table 1 to make skills that are implicit in his condensed list more explicit.

Table 1: Skills deployed within mathematical practices (Based on Kitcher 1984)

These are the ability to:[8]

1. Use the language and symbolism of mathematics;
2. Draw on and use in arguments the accepted statements of mathematics, which include axioms and theorems;
3. Articulate sets of questions and problems regarded as important within the practices of mathematics (and possibly also beyond), both solved and unsolved;
4. Articulate or indicate through examples or represent via other tropes[9] a set of meta-mathematical views including proof and definition standards, claims about the nature, scope and structure of mathematics and its sub-domains, and so on;
5. Articulate or exemplify through examples or indicate via other tropes a set of mathematical values concerning mathematical activities, their products, and their epistemic and ontic status, mathematics as a whole, etc. (a subset of 4);

[8] Skills deployed within mathematical practices are expressed in terms of abilities to use, to draw, to articulate, and so on, to indicate that they comprise observable public manifestations and not some hidden and thus unverifiable mental processes (in line with the Wittgenstein Thesis 6).

[9] By their very nature it is not always possible to communicate meta-mathematical views, values, methods or other aspects of mathematical practice fully and explicitly. In such cases they can be indicated, perhaps incompletely, through other modes of representation such as exemplification in particular cases, models, or canonical representatives of a class, or other linguistic or symbolic tropes such as metaphor, simile, metonymy, synecdoche, parable, diagrams, and so on. This inexplicitness is not surprising for in many if not most cases this is how they been learned, via repeated experience of specific activities, particular cases and exemplars (as is the case generally with tacit knowledge cf. Ryle 1949).

6. Apply a set of accepted methods of derivation, proof and symbolic manipulation based either on rules of derivation and proof or on exemplars and paradigms of proof, as proposed in Kuhn (1970); (This addition to the list is implicit in Kitcher's use of Kuhn.)
7. Operate with and within the constraints of, and to respect, the sets of norms, rules, grammars, accepted methods, standards, regulations of the mathematical practices and mathematical language games as enacted in the mathematical forms of life, whether in the form of explicit rules, or more commonly, norms enacted, accepted and manifested in exemplary and paradigmatic instances of action and behaviour.

The epistemological question (Q1)

With this theoretical framework in place it is now possible to address the epistemological question: How can any social practice philosophy of mathematics provide a foundation for the necessity and certainty of mathematical knowledge? While this question is too general to be answered here in view of the variety of social practice philosophies that have and could be formulated, a specific answer for the particular philosophy of mathematics sketched above follows. Mathematics has a long history of using symbolic formalisms in its calculations and derivations of proofs. Mathematical logic has reached a stage in the past century whereby any mathematical claims or propositions can be represented with greater or lesser satisfactoriness in purely formal terms. The same is true of the logical rules of proof and some mathematical rules too, such as mathematical induction. While there is not quite the same certainty that all mathematical proofs can be so represented, there is a growing body of exemplars where proofs have been rendered in full formalised form, at least in part if not in totality. Plus it is the case that the outcomes of formal rules applied to formal strings of signs within the syntactic domain can be known with certainty.

Overall, we can say that mathematical theorems, proven in the semantic domain gain a great deal of acceptance from mathematicians (in the pragmatic domain). This acceptance is based on the fulfilment of the norms, rules on which the practice of mathematics rests.

But in addition, this acceptability is greatly strengthened by the knowledge that such proofs can most likely be rendered in purely formal terms within the syntactic domain. Once they are so rendered the derivations can be seen as necessary and objective, whose results are known with certainty.

While the derivations are themselves known as correct with certainty, any residual doubts will be restricted to concerns about the correct representation of the semantic proof in purely syntactic terms. Are the strings a valid representations of the mathematical theorems and proofs? Such guarantees and confirmations can only be ascertained with respect to specific instances. Since every rendering of a term, function, proposition or proof in purely sign form has to be tested individually. No general meta-mathematical proof that every mathematical linguistic object can be rendered validly in fully formal sign form is possible. Gödel's (1931) proof with its methodology of arithmetical coding of elementary mathematical terms, functions and proofs provides an optimistic outlook, but it only addresses specific examples.

The social constructionist account provided here rests the certainty of mathematical knowledge on (1) the accepted norms and rules of mathematics, and (2) the possibility of representing mathematical proofs as formal derivations the outcomes of which are known with certainty. Note that the formal rules for strings in the syntactic domain are explicit formal representations or at least explications of norms and rules used in the semantic domain.

So what you have with certainty is that given the starting string, having applied the given explicit rule-based syntactic string transformations, the resultant final string of signs is correctly derived. Assuming the strings and rules are adequate representations in the syntactic realm of their corresponding mathematical objects and rules in the semantic domain, what you have is a formal derivation representing a correct proof that can be asserted with certainty. Thus, with some caveats, we have necessary and objective mathematical knowledge. If the premises are true, and if the rules of deduction are truth preserving, and if their formal representation in the syntactical domain is valid, then the conclusion is also true.

In this account, the norms and rules, which are an assumed and agreed part of mathematical practice, are what underpin the logical and mathematical rules of inference as well as the assumed axioms and suppositions in the semantic domain. In addition they underpin the translations of semantic rules and statements into syntactic procedures and strings of signs in the syntactic domain. These norms and rules can be seen as stipulations, subject to which the correctly proven results of mathematics, its theorems, are known with certainty. This is why mathematics is seen as being inevitable, incontestable and unquestionable. This is how the account of the basis of knowledge is one of necessity rather than contingency. Contingency is pushed back to that of the assumptions and stipulations underpinning mathematical knowledge. No philosophy of mathematics provides such an assurance without assumptions and stipulations. The difference lies in what are the particular assumptions and stipulations in different philosophies of mathematics.

In addition to providing an answer to the epistemological question (Q1) this also begins to answers a further question (Q2) concerning norms and the foundation of mathematical knowledge.

Q2: On the norms and the foundation of mathematical knowledge. If the acceptance of mathematical knowledge and proofs rests on norms what justifies claims of certainty and objectivity, as opposed to contingency?
My answer to this question in many respects matches the traditional response. Subject to the assumption of the axioms and other starting propositions, and the validity of the translation of mathematical propositions and proofs into purely formal systems, the theorems of mathematics knowledge are known as objective with certainty. Contingency is restricted to the choices of axioms and other assumptions, choice of proof methods, and choice of translations into purely formal strings in the syntactic domain.[10] There is also a great deal of confidence in the results of mathematical proofs in the semantic domain. While these cannot deliver truth, they can provide strong proofs of mathematical claims within the semantic domain. Although semantic proofs do not have the full certainty of syntactic derivations, they have one benefit over their more certain syntactic counterparts. Syntactic derivations can be known to be correct. But translating semantic proofs into syntactic derivations are subject to the potential criticism of having their meanings changed by the process of translation into fully formal code, and thus not corresponding fully to their semantics counterparts. Also they may become inordinately long, Your formal derivation may be known with certainty, but is it an anlogue of the proof of what you claim it is? Furthermore, is it surveyable? “A proof must be surveyable; that is, one must be able to take it in as a whole. If you cannot take it in, it does not do what we demand of a mathematical proof.” (Wittgenstein 1978, p. 134).

Mathematical Norms and Rules

Norms and rules enter into mathematical activities through the assumptions of axioms, principles, methods, modes of working, and translations employed in the processes discussed above. Such norms and rules are not justified rationally, by derivation from other norms, rules or assumptions, for this would lead to an infinite regress (Lakatos 1962). Norms and rules are, and must be, justified historically, because of their successful deployment in mathematical practices.

In the earliest cases, mathematical actions such as tallying, counting, drawing shapes with lines, and measuring are part of everyday practices and forms of life

[10] There are, of course, further technical foundational issues such as the incompleteness of all non-trivial first order theories, the impossibility of proving consistency (Gödel 1931), the indefinability of truth. But from the Wittgensteinian position, the security of the foundations of mathematics lies in practices, not in meta-mathematical theorizing and other such safeguards (Thesis 11).

concerned with communication, prescientific observations of the heavens, the seasons and the passage of time, and so on. Such actions may record information pictorially or numerically, often within sacred and religious practices and then more recently; for the purposes of trade, tribute, taxation and management of resources. Mathematics as a discipline started in scribal schools when mathematical knowledge began to be recorded systematically for professional and educational purposes (Høyrup 1994). Without invoking the full history of mathematics the point is that mathematical practices and forms of life had a practical beginning and were a place where the norms and rules of mathematical practice were manifested in practice, and then utilized, demonstrated and taught to apprentice mathematical scribes. Such practices were from earliest times under the auspices of temples and priests so such knowledge was regarded as special and holy. But above all else it was reliable. Computations involving tax, tribute and trade were all regarded with the utmost confidence because their results were invariant and the way that they were derived could be demonstrated line by line by following simple formal rules. Furthermore, they were consistent because different techniques, if correct, deliver the same answers.

Jumping forward to the present, apprentice mathematicians are inducted into research mathematical practices which are where they learn the rules and norms. However, although pure mathematicians and philosophers of mathematics emphasise and indeed often refer exclusively to research practices in mathematics, these are a tiny part of the mathematical forms of life across modern society. Many mathematical practices are not concerned with research but with teaching mathematics in schools and colleges, teaching statistics for scientists and statisticians, teaching mathematical theories to every sort of scientist, plus business administrators, architects, engineers, computer programmers and other users of advanced or service mathematics. There are also special institutions for teaching mathematics to accountants, actuaries, Quants (quantitative analysts), surveyors, and so on; with their own systems of assessment, validation and certification. The norms and rules of mathematics will vary greatly across these practices, which at best share a family resemblance (Wittgenstein Thesis 9). For example, when I studied double entry bookkeeping while working in a business environment it was like nothing I had studied in university mathematics. It had its own unique norms and rules, even though numbers, addition and subtraction, for example, were used equivalently.

In a discussion of mathematical norms and rules there are three distinctions I wish to draw, concerning agreements, different types of norms and levels of norms.

Types of agreement

There are two types of agreement relevant here. The most obvious version of agreement is that reached by overt negotiation between two or more parties where agreed rules are set down. These are explicit and also have explicit approval when part of an agreement. Some of mathematical assumptions are of this type, such as the agreement to use the axiom of choice in a specific set of researches into axiomatic set theory.

The second type of agreement is closer in meaning to the way the term is used in engineering. Here agreement between machine components means that they run closely together to a high degree of accuracy. In an efficient petrol powered automobile there will be close agreement between the pistons and cylinders in the engine. Agreement of this second type means running in parallel, with closely matching actions. Many agreements in mathematical practice or research are of this type, with the mathematicians' procedures and actions running smoothly in parallel, well aligned, without overt or explicit specification of the rules and norms that they are both employing. Since many of the norms and rules of mathematical practice are implicit in the way the practice is conducted, and have not been made fully explicit, this is the way that agreement in shared practice must be enacted. That is, through parallel practices, in which one or the other participant can guide or steer the other by suggesting minor changes in direction, but where the underlying norms and rules can only be exemplified; cannot made fully explicit. These are agreements in practice; alignments of practice.

When we speak of agreements in social practices as the basis for mathematical activity, it is very often this second type of parallel, aligned or comparable, perhaps even shared types of activity, rather than explicitly negotiated or imposed agreements. They are not the types of agreements made in overt pacts, contracts or as a consequence of explicit negotiation or bargaining.

Entry into any language game within a form of life, a social practice, means that the agreements within it stem from a shared history of the actors within that form of life, and during that period of engagement the actors practices are brought into alignment, into agreement from minor 'nudges' or corrections from senior participants addressed to apprentices and junior participants (but not exclusively in this direction). It is through such interactions that the norms and rules of the language game and form of life are communicated, sustained, enacted, their uses corrected, and possibly even have their scope extended.

The agreement in admitting a new practitioner into a social practice is that they will take on its roles, practices and goals. Such terms of membership may be fully explicit, even though the norms and rules are not. However, the terms of membership can also be implicit in the act of participation in the practice. When you join a practice it is assumed you will respect its norms, rules and goals. The workaday norms and rules of the form of life are 'learned' or 'acquired', that is observed via instantiation, and also copied, respected, navigated and enacted, through extended participation in the activity (Lave and Wenger 1991).

Thus when one claims that social constructionism rests on agreements on the norms and rules implicit in the institution of research mathematics, one is not claiming that participants they have signed a fully explicit contract, or its informal equivalent, with full agreements to all aspects of the practice. Their agreement means that when they are seasoned participants within the practice their actions will respect, embody and represent, through particular instances, the norms and rules enacted within the practice. A participant's individual practices will be aligned with those of other group members. Furthermore, this is how the practices are sustained, through new members aligning with the norms as apprentices, maintaining them as journeymen, teaching them as masters, and all together guiding the practice into the future.

Types of Mathematical Norms

Since the norms of mathematics play such a central role in the philosophical account of mathematics provided by Wittgenstein and also that offered here, it is best to defined them more clearly. There are many norms in the practice of research mathematics. These pertain to the following.[11]

(1) **Social relations**. Interpersonal relationships within the social practice, including the acceptance of and acknowledgement of power differentials and authority, such as those manifested in gate-keeping and the guiding of apprentices;

(2) **Goals.** The type and choice of problems addressed and theorems proved, both in terms of their significance and how they relate to the broader field of mathematics;

(3) **Style.** The rhetoric of written mathematics in the particular specialism, how papers should be structured and organised as linguistic productions (Knuth et al. 1989);

(4) **Content.** The nature, content and form of the proofs, including references to other results, the adequacy of definitions, and the levels of explanation, proof steps and jumps in reasoning that are acceptable;

(5) **Correctness.** The identification of correct and incorrect proof steps, and of adequate or inadequate translations of mathematical expressions and proof derivations into purely syntactic sign sequences.

(6) **Existence.** The elements of mathematics, including numbers, sets, functions, functors, operators, structures, models, theories, etc. are all treated as if they exist independently of mathematicians. The linguistic

[11] This is offered as a provisional working list rather than a definitive one.

objects of mathematics are also treated as objects of mathematics, or as the names of mathematical objects (terms, functions, etc) or as relations between mathematical objects (sentences, expressions, theories, etc.).

The first three are located primarily in the pragmatic domain of mathematics, although they all constrain activities in the semantic domain. The fourth, fifth and sixth primarily concern the semantic domain, but the fifth also regulates what is permissible in the syntactic domain. In this case the rules are explicit, both how to formulate strings correctly and how to transform them to derive new strings. Apart from the application of norms of type (5) to the syntactic domain, none of the rules or norms of the practice in research mathematics can ever be made fully explicit. There is no supervening explicit meta-norm that can determine all of the cases exhibited to date, let alone the as yet unspecified instances and exemplars of norms in practices (Wittgenstein Thesis 12).

The concrete manifestations of these norms in both exemplars and critical feedback is what shapes and regulates the nature of new published knowledge contributions. The norms of justifiability and acceptability, that serve as proxies for mathematical certainty and near-certainty, are enforced by knowledge gatekeepers (supervisors, senior mathematicians, referees and journal editors) in their judgements of submissions. Because of the dialectical process of validation involved, the emergence of new mathematical knowledge contributions, including judgements of correct formation and the acceptability judgements, follows a conversational, back and forth, pattern (Ernest, 1998, 2023, Dutilh Novaes 2021). One product of these social processes is that most responses to texts in mathematics by experienced mathematicians are in agreement with the norms and rules of the practice. Mastery of the language games shapes these responses of mathematicians in agreement with the traditions and norms of the practice, the particular mathematical form of life they inhabit.

Deep versus Surface Norms

One of the criticisms of the view that mathematical knowledge and certainty rests on norms is that they appear contingent, changeable, insubstantial and even arbitrary (Dieterle 2010). If norms are decided by people's choices, or so it can be argued, then they could be changed on a whim. Or powerful forces could change them willy-nilly. For example, in a dictatorship it could be decided to dictate that 7 is not a prime number but an even one, because 7 is views as a lucky number by the dictator. This would have a large impact on number theory. So many exceptions and more complicated definitions would have to be added and it might render number theory unusable.

In order to combat the idea that norms can be changed on a whim, I need to distinguish two different levels of norms, deep and surface norms (Ernest 2023).

Logic based on deep norms is very different from, say, chess based on surface norms. Within a deductive theory based on logic there are logical connections between many assertions which mean that individual axioms, rules or truths cannot be changed without threatening the stability, consistency and identity of the whole system. This differs from chess where we could allow a pawn to jump up to 3 squares at first, or let kings move two squares at a time without any contradictions or destabilization of the game. This exemplifies the difference between deep norms (those of logic and mathematics) and surface norms (those of chess and other games and contingent practices) whose alterations only affect the practice locally and superficially, without deep implications throughout the whole practice. Furthermore "Unlike the rules of chess, the rules of mathematics have important applications outside their own domain" (Pantsar 2021, p. 323). However we change the norms and rules on mathematics they must remain compatible with well established extra-mathematical uses and applications

The deep norms, found in logic and mathematics, are richly interconnected by logical inferences and dependencies, as well as conceptual definitions. Such norms are deeply embedded in mathematical practice and mathematical traditions and no one of them can be changed without risk to the stability and consistency of the whole system, and its applications.

Although, following Wittgenstein avoiding contradiction (Thesis 11) is not the be all and end all of mathematics, the univocity of functions and processes does remain vitally and inescapably important, for without it the necessity that arises from explicit rule following would be compromised, and representation in the syntactic domain would not be possible.[12] Indeed the very definition of a function requires a unique output for every input, or it is inadequate. One of the safeguards of mathematical knowledge and certainty is that its proofs can in principle be translated into formal derivations in the syntactic domain. Univocityis essential, a *sine qua non*, for this to be done. The designation of Two by '2' is a surface norm, and different signs are used for Two in different cultures ('dos', 'tva', 'II', etc). But the fact that Two is the successor of One, however, they are represented, is a deep norm adopted as a fact. The measure of the sum of the angles of a triangles as 180º is a surface norm, although it has implications for how all other angle measures are denoted across geometry and society. But the fact that this sum is two right angles in plane geometry is a deep norm within Euclidean geometry. Tamper with this and the system goes awry.

Deep norms are mainly the fourth and fifth type of norm identified above, the norms behind the identification of correct and incorrect steps in a proof, and

[12] This does not compromise Wittgenstein's argument that implicit rule following is open to interpretation. While there are many ways to continue the sequence 2, 4, 6, 8, ... there is only one answer for each input n in $f(n) = 2n$, thus resulting in a unique sequence beginning 2, 4, 6, 8, ...

correct definitions. Mathematics is deeply conservative, and axioms, definitions, rules of inference, theories, and applications are maintained more or less invariant within mathematical systems and domains.[13] Some of these elements may be refined or made more precise, but unless a new theory is being formed, they will be conserved and maintained, thus also preserving applications (Pantsar 2021). Such is the scope of deep norms.

The contrasting surface norms include the rules of chess, elements of any ritual, style or fashion; also laws, legal injunctions, club and sports rules, classroom rules, and also the social rules of conduct in mathematical practices. These may be firmly enforced in their settings, but one can change elements of them without jeopardising the whole practice. The rules of tennis and football serve as good examples. These have been changed significantly over my lifetime to suit outside interests, but the games function as well as they ever did. However, the norms of logic and mathematics are deep norms which means that their necessity is well buttressed by practice and challenging them has far greater significance. Such norms are deeply embedded in the practices, expectations and traditions of mathematicians who strenuously act to preserve them. Thus, such notions as proof and truth, although based on norms, are largely preserved over time, although they do shift, little by little, over the course of history (Hersh, 1997; Kline, 1980; Lakatos, 1976).

Indeed what Gillies (1992) refers to as ‘revolutions in mathematics’ can occur when norms are changed, usually meta-mathematical norms, such as the nineteenth century shift from intuitive theories to formal theories. This involved concomitant shifts in meanings and interpretations from a single intended model for each theory to multiple models. For example, *The Frege-Hilbert controversy* at the end of the 19th century was centred on the legitimacy of this move in the interpretation of geometry from the classical notion of Euclidean intended meanings to axiomatic theories where terms are reinterpretable placeholders supporting many models (Hallett 2010).

It is possible to challenge and reject some deep norms, but the result is to make a new theory rather than to challenge an existing theory. Hamilton rejected the property of symmetry in algebra ($a \times b = b \times a$ is rejected) to invent the new Theory of Quaternions (Pickering 1995) in which $i \times j = {}^{-}j \times i$. Brouwer (1913) and Heyting (1956) rejected the logical axioms of double negation ($\neg\neg A = A$; not-not-A is equivalent to A) and Excluded middle ($A \vee \neg A$; either A or $\neg A$ is true). In consequence they created a new mathematics, Intuitionistic mathematics, in which many of the results or classical mathematics are untrue and unprovable.

[13] Of course in the formation of a new domain, such as topology, a complex dialectic is at work historically to define concepts and prove theorems in which meanings and results are not stable, as Lakatos (1976) amply shows. But the mature theory will exhibit stability and even invariance once canonically formulated.

Intuitionistic mathematics represents a radical break from classical mathematics, developing in its own original direction by dint of rejection of a couple of laws of logic.

Having said this most if not all results of Intuitionistic mathematics are acceptable in classical mathematics, although some of their results and definitions are not of interest to classical mathematicians, such as the Bar and Fan Theorems, and the definition of choice sequence. The last of these leads to the classically unacceptable result that “Every total function on the continuum is continuous.” (Moschovakis 2017, §3.4)

Overall, mathematical necessity is learned as a product of deep norms applied over time so as to warrant specific mathematical knowledge productions as correct, ‘true’ and necessary.[14] Its definition is extensional, given by exemplars warranted as necessary, rather than intensional, given by explicit statements of definition. As an abstract category the concept of necessity can be wielded by metaphysicians as if it has a solid, if abstract, presence. In practice, it is determined by a myriad of material interactions drawing on persons’ judgements and their outcomes. These are an essential part of the social construction of mathematical knowledge, drawing on human conversation (Ernest, 1998, 2023, Dutilh Novaes 2021). This human fabrication, as Vico (1710) argues, is the only one we can know with certainty. It is the sole embodiment of necessity, resting as it does on our own rules, norms, language games, forms of life and lived epistemic practices. Mathematics is, as Vico suggests and John Dewey affirms, is a figment of man’s creative intellect (Dewey 1920).

Q3. The ontological question

With respect to the existence of mathematical objects, how is their seeming and widely accepted objectivity, reality and existence accounted for and vouchsafed within a social practice philosophy of mathematics so as to guarantee that they are substantial, tangible and virtually palpable? How does it avoid subjectivism and nominalism without slipping into Platonism?

This is the most difficult question to answer, because the social account of mathematical objects is based on change and development. The natures of the objects of mathematics change greatly and significantly both over the historical development of mathematics as well as during the development of the individual mathematician.

[14] ‘Truth’ in mathematics is always at best provisional, and is usually represented by ‘provable’, which is known to fall short of truth.

This contrasts with the main traditions of Western philosophy, which treat universals, abstractions and the objects of mathematics as existing in the timeless present. From this perspective, the objects of mathematics and ontology more generally are seen as unaffected by time and space, since they concern only the unchanging essences of things. For confirmation of this view see, for example, the relevant works of Plato, Frege and Gödel, as well as Balaguer (1995).

However, Wittgenstein comes out strongly against Platonism. He claims that mathematical statements need not be understood as referring to mathematical objects or a mathematical reality (Thesis 10). Wittgenstein also asserts that mathematical statements are not descriptions of pre-existing objects and relations (Thesis 3), and that logic and mathematical formalisms are techniques, not foundations (Thesis 11). So much for his negative ontological stance.

In contrast his positive ontological views are that the meanings and uses of mathematical ideas and objects grow and change and so at best they share family resemblances and there no fixed essences in mathematics (Thesis 9). Furthermore Proofs and the derivations they depend on concept-evolving acts (Thesis 7). Thus all mathematical objects including the meanings of terms, concepts, propositions, proofs, theories, and even mathematics itself as a body of knowledge and as a social practice are in states of becoming, changing, growing and development.[15] Lakatos (1976) demonstrates this in historical case studies and goes on to establish a dialectical philosophy of mathematics drawing on the theories and methods of Hegel, Polya and Popper (Tanswell et al. 2025).

Whitehead (1978) applies his process metaphysics - especially the concept of becoming - to mathematical objects. He reinterprets mathematics as grounded not in eternal, static entities, but in processes, relations, and patterns of becoming.

> "The doctrine that the world is composed of timeless objects eternally existing in a separate realm is entirely misconceived. Mathematics derives its meaning from the world of process." (Whitehead, 1929/1978, p. 12)

Whitehead see mathematical objects as ideal structures abstracted from the patterns of process in the world: "Mathematics is the study of pattern. This pattern is derived from the world and applied to it." (Whitehead, 1925, p. 17). "Mathematics is nothing else than the description of the general forms of pattern." (1925 p. 39).

[15] Such change is non-teleological, with no fixed or pre-ordained endpoint to the developmental process.

From the perspective of this paper there are two main weaknesses in Whitehead's account. First, he does not describe in any detail how the processes in mathematics are abstracted into structures and mathematical objects. Second, his philosophy is too empiricist. It can be interpreted as saying that mathematics is made up of patterns abstracted from the world and then applied to it. In fact the life of mathematics as an autonomous practice, with thousands of years of cultural and historical development, intervenes between the abstraction of ideas and patterns from the world of experience, and the applications of mathematics. In between lie the complex layers of abstraction, reorganization, theory construction, and yet further abstraction, within the world of pure mathematics, before mathematical models made ready to be applied to the world.

Roman Ingarden (1981) develops an ontology that explores the distinction between Intentional objects: objects constituted in acts of consciousness (including mathematical entities), and ideal objects: which exist independently of acts of consciousness. Ingarden's position suggests that mathematical objects originate as intentional objects through acts of idealization, and can come to be treated *as if* they were autonomous and independent through communal and methodological practice.

Barry Smith elaborates on this:

> "The reification of intentional objects occurs through a layering of acts—first individual, then communal—which result in the stabilization of meanings and the apparent objectivity of ideal constructs such as those of mathematics." (Smith 1981, p. xlix)
>
> "These entities do not *exist* in the way spatio-temporal objects do, but acquire a quasi-objective status through intersubjective constitution." (Smith 1981, p. xlv)

Smith suggests that mathematical objects are "made" through demarcatory cognitive acts**,** which over time and communal practice enable the creation of mathematical theories which, together with the objects of mathematical acquire a status akin to autonomy. (Smith 1975).

Linnebo (2018) claims that many **a**bstract objects in mathematics are introduced by abstraction principles. Thus, for each equivalence class under a salient relation there is a corresponding object. This is the mechanism whereby mathematics 'makes' its own entities.

Mancosu (2016) analyses abstraction principles as formal mechanisms that construct mathematical objects from equivalence classes, making them real in virtue of grammar. He argues that mathematical entities are introduced via definitions by abstraction; effectively they are reified constructions.

Kitcher (1984) explicitly characterizes mathematical objects as emerging from idealized operations, with constructed practices turned into abstract objects. Thus collections and numbers are reified abstractions - not self-standing entities, but formalizations of communal manipulations.

Machover (1983) argues that the objects of mathematics are formed as 'reified constructions'. He is the first philosopher of mathematics to explicitly use this term.

More detailed work has been done on this in cognitive science and mathematics education. Thus Anna Sfard describes the turning of processes into objects as reification in the context of learning mathematics. She outlines three stages in concept formation: from interiorisation via condensation to reification, describing exactly how processes become objects in math cognition.

1. **Interiorisation**: Automatising the procedure, e.g., counting.
2. **Condensation**: Treating the process as a whole, e.g., viewing the operation of "adding" as a whole, rather than a sequence of actions (how to add).
3. **Reification**: Viewing the procedure as an abstract entity, e.g., the number "3" as an object (a cardinal number), not just the outcome of the act of counting three steps ending in "third" (an ordinal number).
   "Reification, therefore, is defined as an ontological shift – a sudden ability to see something familiar in a totally new light. … The new entity is soon detached from the process which produced it" (Sfard, 1991, p. 20)

This is the genetic mechanism that creates abstract entities (e.g., negative numbers, functions), and provides a description of how mathematical objects emerge from practice, notation, and discourse. Furthermore, this process is cumulative with increasing levels of abstraction, as actions on simpler objects become reified into more complex objects in themselves, recursively.

The process of reification is also called objectification, entification, encapsulation (Dubinsky 1991), and the formation of *procepts.* A "procept is an amalgam of concept and process represented by the same symbol." (Gray and Tall, 1994, p. 116). It also parallels the process of nominalisation in linguistics, whereby a verb form (action) is turned into a noun (name of an object). Note that nominalisation is a natural and universal process that occurs during the use of language (Halliday and Matthiessen 2004). The claim here is that the reification of actions and constructions is similarly universal in the learning, use and formation of mathematics, both by individuals and by cultures.

So now I can answer Question 3: *Why are mathematical objects objective?* Mathematical objects are objective because at any given time they appear 'solid' that is with some kind of identity and substance that is invariant, resistant to change and with enduring properties, that are not eternal but long lasting. They are founded on mathematicians' norms, rules and agreements. They receive support for their enduring existence because of their fixed, shared uses in the domains of mathematical practice, and beyond in publicly shared uses in everyday and real world applications (Pantsar 2021). Their uses are rule governed and there is widespread agreement without ambiguity as to correct usage within mathematics and beyond.

Once created mathematical objects "detach from their originator" (Hersh 1997, p. 16) becoming independent and self-subsistent entities within a shared domain, the cultural space of mathematics (Ferreirós 2023). Their sustained and agreed use normalises them as accepted cultural objects, and this acceptance and use warrants their existence as independent from particular uses and individual mathematicians (Barnes 1982).

However, if mathematical practices shift over time, so too may both the rules and objects of mathematics themselves. Thus, in reflecting such cultural shifts, they are dynamic not static objects and practices, although in actuality virtually all of their properties are conserved and they are slow to change. Ontologically they are regarded and treated as permanent and enduring entities that exist objectively even though they do not exist outside of human cultures and practices (Ernest 2023). Their objectivity is social, and they are not a given part of any ultimate reality, should such exist (Bloor 1984).

Q4. The practice question: Can an account of mathematical practice be given that fits with mathematicians' experiences of doing mathematics as practitioners of the art?

How does a social practice philosophy of mathematics based on Wittgenstein's ideas and tenets provide a foundation for beliefs in the necessity and universality of mathematical knowledge, and in the objective existence of mathematical objects? What overall view of mathematics supports these beliefs?

The pragmatic domain of mathematics is the human and cultural domain that contains but exceeds the syntactic and semantic domains of mathematics. It includes the norms, rules, beliefs, values, goals and philosophies of mathematics as embodied in mathematical practice and its language games (including meta-mathematical discourse). It also encompasses all of their mathematics-related actions as well as the overall social milieu of the mathematical form(s) of life.

For the purposes of answering Question 4, I shall concentrate on the beliefs, values and goals of mathematical practice that need to be addressed in a social philosophy of mathematics for it to be, at least potentially, satisfactory for a working research mathematician. These are primarily ontic and epistemic beliefs and values, although as values they are part of the concerns of axiology, and of neither epistemology nor ontology. I shall leave aside aesthetic and methodological values as these are compatible with many different philosophies of mathematics, and are not central to the argument being made here,

Epistemic beliefs and values of mathematicians

Mathematicians express their beliefs in the certainty, necessity, and truth of mathematics powerfully. These are part of their meta-mathematical views and beliefs about mathematics, within the category Kitcher proposes (1984).[16]

Bertrand Russell emphasized its truth and beauty. “Mathematics, rightly viewed, possesses not only truth, but supreme beauty” (Russell 1991, p. 368)

Ian Stewart argues mathematical truths and their supporting proofs “have a permanence and certainty about them that other ideas in science lack. This permanence is an endearing feature of our discipline” (Stewart 2002, p. 195). He underscores the exceptional assurance and stability afforded by mathematical proof.

Gödel emphasizes the belief in the objective, eternal existence of mathematical truths beyond human opinion. "It is the conviction of all mathematicians that the truths of mathematics are objective and independent of human thought, existing eternally in a realm accessible through proof." (Gödel, 1931, p. 18).

Marcus du Sautoy strongly affirms the stability and permanence of mathematical knowledge grounded in proof. “Mathematics is a universal language with an unshakeable foundation, built on proofs that offer an enduring certainty." (du Sautoy, 2015, p. 88).

Tim Gowers expresses the assuredness and eternal validity of mathematical proofs. "The beauty of mathematics lies in its certainty - once a proof is

[16] Although indicated here as beliefs of individual mathematicians, my understanding is that such beliefs and goals are social, part of the axiological commitments that are an integral part of membership in mathematical practice and learned alongside the rest of the working knowledge and methods of mathematics.

established, it stands forever, providing an unshakeable foundation of truth." (Gowers, 2014, p. 142).

Roger Penrose echoes a belief in the enduring and fundamental nature of mathematical truth and its ontological basis. "Mathematics is not merely a language that describes the universe; it is a fundamental part of the universe itself, with an eternal validity that underpins physical reality." (Penrose, 2005, p. 251).

Many other mathematicians could be cited such as Ingrid Daubechies "Mathematical results, once proved, become part of an eternal, unchanging landscape of knowledge." (Daubechies, 2012, p. 34), and John Stillwell "Mathematics offers a universe of eternal truths, unaltered by time, that stand as a testament to the certainty of human reasoning." (Stillwell, 2010, p. 212).

What these quotations indicate is the widespread belief in the enduring, certain, and universal nature of mathematical knowledge and truths, reinforced by the rigor and stability of proof.

The account given earlier in answering the epistemological question (Question 1 above) is that the social practice philosophy of mathematics, as explicated here, provides a very strong foundation for these beliefs. The rules and norms on which mathematical knowledge rests are firm and deep norms that offer an enduring and stable foundation for mathematics. Furthermore, the acknowledged certainty of the sample derivations encoded within the syntactics of mathematics, and the belief that all mathematical proofs could be so encoded, provides even firmer foundation, as does the enduring use and validity of mathematical applications. Mathematics not only rests on firmly entrenched social rules and norms. It also has examples of mathematical knowledge that have been established with certainty. Irrespective of whether mathematical knowledge is absolute, timeless and true (and my view is not) there are excellent reasons for a belief in this character. It remains the firmest knowledge attained by humanity, thus justifying mathematicians' belief in it.

What can be said is that the aspiration to truth and certainty represents one of the most deeply held values of mathematical practice. Mathematicians are happy to leave the epistemological niceties to philosophers, as long as they can maintain their beliefs in, and goal of striving towards, mathematical certainty. They can safely continue to believe that they are aspiring to the truth in mathematics. Although not providing a warrant for this, the success of mathematical applications adds a further layer of confidence in the certainty and truth of mathematical knowledge.

However, this analysis means that truth and certainty are not necessarily epistemological attributes of mathematical knowledge but are instead axiological. They are values to be aspired to rather than attained metaphysical facts. Truth is anyway unattainable in mathematics, for all mathematical results depend on assumptions.[17] Only when these are known to be true, which they are not[18], can we claim that any conclusions reached correctly are true. But to be proven within mathematics remains a good proxy for truth in mathematics, one that allows mathematicians to continue in their belief in truth and knowledge, and that it is known with certainty.

Ontic beliefs and values of mathematicians

The widespread adherence to Platonism by mathematicians represents a belief in the objective and independent existence of mathematical objects (Cohen, 1971). Working mathematicians have strong beliefs in the independent existence of mathematical objects, mathematics itself, and see their discoveries and creations as potentially timeless and eternal.

G. H. Hardy argues for hard mathematical permanence. “Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not.” (Hardy 1992, p. 23).

John Horton Conway sees mathematics as a lived, exploratory world. “How we can apprehend this amazing world that appears to be there … No mathematician believes that the mathematical world is invented. We all believe it’s discovered. That implies … there is an ideal world.” (Conway 2001, p. 9).

Charles Hermite also sees mathematics as a realm independent of human existence.

> “There exists, if I am not mistaken, an entire world which is the totality of mathematical truths, to which we have access only with our mind, just as a world of physical reality exists, the one like the other independent of ourselves, both of divine creation.” (Darboux 1906: p. 46)

But several mathematicians including Paul Cohen (1971), although Platonists at heart don’t feel they should to admit to this view in public; they feel it’s not respectable.

> “The typical working mathematician is a Platonist on weekdays and a formalist on Sundays. That is, when he is doing mathematics he is

[17] It is a well establish position, since the time of C. S. Peirce, that mathematics is an hypothetical science (Cantù and Testa 2023).

[18] Nor are they false. They are simply assumptions.

> convinced that he is dealing with an objective reality whose properties he is attempting to determine. But then, when challenged to give a philosophical account of this reality, he finds it easiest to pretend that he does not believe in it after all." (Davis and Hersh, 1983, p. 321)

According to Lipman Bers:

> "A working mathematician is always a platonist. It doesn't matter what he says. He may not be a platonist at other times. But I think that in mathematics he always has that feeling of discovery." (Albers, et al. 1990, pg. 19).

This highlights the dual stance that mathematicians hold: practising with a belief in objective mathematical reality, yet adopting formalist rhetoric when pressed philosophically.

Koopman argues that the belief in the real existence of mathematical objects can coexist with social constructionist philosophy of mathematics.

> "Mathematics is a product of human activity that acquires a status and a structure, sustained by the practices of scientific and educational institutions, such that its objects and concepts resist denial. They are culturally real, not metaphysically real." (Koopman, 2004, p. 66)

Thus such beliefs are compatible with the social activity philosophy of mathematics based on Wittgenstein that is sketched in this paper.

Whatever the expressed philosophy, mathematics possesses or takes on a life of its own. "This body of truth is seen as existing in its own right independently of whether anyone believes or even knows about it." (Bloor, 1973, p. 43,)

Michael Atiyah vividly expresses the autonomy of mathematical forms, existing in an ideal realm that is of profound significance - even though not instantiated in the physical world. "Perfect spheres do not exist in the real world, but they do have reality. They exist in the human imagination - and that's the most important world there is." (Atiyah, 2016, item 15)

This opinion is echoed by Grothendieck "Mathematics is not a universe created by us; it is a realm that exists in itself, with its own laws and logic." (Grothendieck, 1985, p. 9). "The mathematical objects are not our inventions; they dwell in a certain realm, alive and waiting, like a secret garden that we gradually discover, piece by piece." (Grothendieck, 1985, p. 112)

Ramanujan claims that not only does it exist in a realm beyond us, but that mathematics is divine thought. “An equation means nothing to me unless it expresses a thought of God.” (Ramanujan 1927, p. xxiii).

Novalis also attributes an elevated, quasi-religious nature to mathematics.

> *“Mathematics is the life supreme. The life of the gods is mathematics. All divine messengers are mathematicians. Pure mathematics is religion. Its attainment requires a theophany.”* (Novalis 1901, p. 223)

This poetic depiction endows mathematics with divine vitality and transcendence, effectively personifying it as a living, sacred force. This is echoed by Bragdon *“Mathematics is the handwriting on the human consciousness of the very Spirit of Life itself.” (*Bragdon 2005, p. 61). This depiction portrays mathematics as an organic imprint of life, as something flowing through us.

As many mathematicians testify, mathematics itself takes on an autonomy apart from its creators,. "Mathematics is not a mere tool but a living organism. Its definitions breathe, its theorems evolve, and its structures grow according to inner laws we merely glimpse." (Poincaré 1908, p. 29). "A mathematical object is not invented—it is encountered. Once defined, it acquires a life of its own, demanding we follow its logic rather than impose ours." (Rota 1997, p. 148).

The idea of mathematics as force a that takes hold of a mathematician and flows through them is expressed more fully by Andre Weil.

> “Every mathematician worthy of the name has experienced … the state of lucid exaltation in which one thought succeeds another as if miraculously … this feeling may last for hours at a time, even for days. Once you have experienced it, you are eager to repeat it but unable to do it at will, unless perhaps by dogged work.” (Weil 1992, p. 91

Imre Lakatos takes a further step in attributing autonomy to mathematics and seeing it as taking over and flowing through mathematicians. Thus mathematics is an entity that although created by humans is experienced as something external, with a life of its own that controls the mathematician and is expressed through their activity.

> Mathematics, this product of human activity, alienates itself from the human activity which has been producing it. It becomes a living, growing organism, that acquires a certain autonomy from the activity which has produced it, that develops its own autonomous laws of growth, its own

> dialectic. The genuine creative mathematician is just a personification, an incarnation of these laws which can only realize themselves in human action. Their incarnation, however, is rarely perfect. The activity of human mathematicians, as it appears in history, is only a fumbling realisation of the superb dialectic of mathematical ideas. But any worthwhile mathematician feels the sweep of it, and obeys, if he has talent, spark, genius, i.e., if he can communicate with this dialectic of ideas. (Lakatos 1961, p. 178)

It is this feeling, this experience of the creative mathematician that recognises mathematics as a living entity, with a life of its own, an outside force whose presence is felt most strongly and palpably during the act of creation. This is experienced as an act of discovery, the exploration of a pre-existing landscape, with its own mountains, valleys, lakes and boulders, which are the objects of mathematics. Such experiences validate the belief in Platonism, in the independent existence of all of the ideas, results, theories; the objects of mathematics. Any philosophy that denies this, that claims mathematics is just made up of formal symbols and their games, is just logical reasoning, or is just a part of culture, fails to recognise or acknowledge the lived experience of mathematicians. This experience is of mathematics as an entity with a life of its own which flows through the mathematician in the exalted state of creation. A belief in the reality of mathematics is intrinsic to the creative mathematician.

Any philosophy of mathematics that seeks to capture the lived experience of the mathematicians must make a place for this belief. But not necessarily in adopting a formal Platonist stance, but in acknowledging the centrality of the values of independence, objectivity and autonomy of mathematics and its objects. But these are ontic *values*, a set of beliefs and practices that underpin creativity in mathematics. They are to be located in an axiology of mathematics, not its ontology. Seeing mathematics as a powerful entity that exists outside us but possesses and haunts mathematicians may be quasi-religious, but it is part of mathematicians' lived experience and deeply embedded in their metamathematical beliefs and values.

Traditional arguments about the ontology of mathematics have failed to recognise this. Harré and Krausz (1996) are among the very few to treat independence and objectivity as ontological values, as a part of axiology and not as ontological facts, when applied to mathematics and elsewhere.

Viewing Lakatos' paragraph quoted above as a statement of the ethics or ethos of mathematics fits with the MacIntyrean virtue ethics position (MacIntyre 1984). This asserts that the good of a practice is its inner purpose, its direction and goal. What drives mathematicians is the living dialectic of mathematics, as a good. This mysterious constructed entity we call mathematics is the crystallization into being of the substance of mathematics but it also represents the goods, values and the target of mathematical activity. Approaching mathematics, working to understand, make and further extend it represents and embodies, is the good that mathematicians strive for. Serving this body, this externalisation of the spirit of mathematics, is the ethics of mathematics, the ethos of mathematics, the good of mathematics.

This is what mathematicians can recognise and identify with. Mathematics lives outside them (yet through them) and demands their loyalty and submission. Only initiates can feel and know this. This vision is central to the practice of research mathematics. But it is virtue-ethical, or axiological, not epistemological nor ontological, to us outside commentators.

Thus the answer to Question 4 is that the certainty of mathematical knowledge, and the independence and objectivity of mathematical objects are acknowledged as working mathematicians require, in a social practice philosophy of mathematics. But they are acknowledged as epistemic and ontic beliefs, values and goods, and thus as part of the axiology of research mathematicians' practices. The traditional assumption that they are epistemological and ontological facts is unnecessary and is repudiated. Provided that we address mathematicians on working days, when they are immersed in the practices and their associated beliefs and values, and not on Sundays, when they put their best philosophical clothes on, they should be satisfied.

Scholars have confirmed this eschewing, this avoidance of traditional philosophies of mathematics by mathematicians.

> "Mathematicians rarely read papers in the philosophy of mathematics. The reason is simple: Most philosophers of mathematics are not mathematicians, and their writings are usually irrelevant to mathematical research." (Nathanson 2008, p. 1).
>
> By far the larger part of activity in what goes by the name *philosophy of mathematics* is dead to what mathematicians think and have thought (Corfield 2003, p. 5)

Jean Dieudonné offers a mathematician's reflections on philosophy and mathematical practice.

> "On foundations we believe in the reality of mathematics, but of course when philosophers attack us with their paradoxes we rush to hide behind formalism and say, 'Mathematics is just a combination of meaningless

> symbols,' … Finally we are left in peace to go back to our mathematics and do it as we have always done, with the feeling each mathematician has that he is working with something real. (Jean Dieudonné quoted in Hersh 1979, p. 32)

Gian-Carlo Rota (1997) also affirms that philosophical doubts about the nature of mathematical truth are of little concern to working mathematicians. Likewise concerning ontology "The existence of mathematical items is a chapter in the philosophy of mathematics that is devoid of consequence" (Rota 1997: p. 98). He argues that mathematicians need results, not metaphysical justifications. In this way, contemporary philosophy of mathematics is devoid of interest for mathematicians.

What I have endeavoured to do here is to support mathematicians' beliefs in the objectivity and certainty of mathematics. Mathematics, as a body of knowledge practices and the objects worked, with does exist autonomously outside of any individual or small group of mathematicians. For, "it seems, for humans, platonistic thinking is the most efficient way of working with mathematical structures." (Podnieks 2015, p. 3).

But this objectivity is social and is a goal rather than a fact. "[W]hile the mathematicians investigate the formal realm independent of them, the mathematical discoveries take place in mathematical practice and hence are socially constructed." (Hartimo and Rytilä 2023).

This is neither magic nor alchemy (Thesis 10). The objectivity is a part of the ethos and ethics of research mathematics, perhaps even of its heuristics. It is axiological rather than an ontological and metaphysical fact. Likewise, with the belief in the certainty and timeless universality of mathematical knowledge. This is not to say that there is no certainty in mathematics, as I have argued. But that believing in the absolute certainty of the whole of mathematics is just a shared belief, an aspiration, a good that is part of the ethos and ethics of research mathematics. It is axiological rather than epistemological. It is something mathematicians strive for, even if they can never hope to fully achieve it.

Conclusion

This paper attempts two main tasks. First to elucidate the key tenets of Wittgenstein's philosophy of mathematics, and to extend them into a social constructionist and social practice based philosophy of mathematics. Second to test if this proposed philosophy of mathematics can satisfactorily answer four questions as to its adequacy. These concern first, the epistemology of mathematics and, in particular, the certainty of mathematical knowledge;

Second, the epistemic and other roles of rules and norms in mathematical practice; Third, the ontology of mathematics; what are mathematical objects and how far can we accommodate their objectivity? Fourth, how far does the proposed social practice philosophy of mathematics go in answering the concerns of practising research mathematicians to their satisfaction?

In order to address these areas in depth a semiotic analysis of the fundamental sign practices of mathematics is employed methodologically. This distinguishes between the syntactics, semantics and pragmatics of mathematical practice. The syntactic domain allows for the certainty of uninterpreted mathematical derivations. Although these are not fully fledged proofs (**Thesis 7)**, they do constitute the skeletons, the formal underpinnings of proofs, when interpreted in the semantic domain (and accepted in the pragmatic domain). The semantic domain is the home of the actual proofs published by mathematicians and admits numerical and truth-values into the language games of mathematical practice,

The pragmatic domain with its human and social aspects of mathematical practice admits meta-mathematical views, values and the philosophies of mathematics subscribed to by mathematicians (and others). It is within this domain that the epistemic and ontic values and goals of mathematics are located. It is claimed the proposed social philosophy of mathematics accommodates these ontic and epistemic values and the goods of mathematical practice located there, that are held dear by pure mathematicians.

Because the social philosophy of mathematics proposed here is social constructionist, it rejects the claims that mathematical knowledge is absolutely universally and metaphysically true, and that mathematical objects are fully objective existing outside of our material world and cultures in some other domain. However, it does accommodate the epistemic certainty of mathematical knowledge and the ontic independence of mathematical objects as beliefs, values, goals and goods of mathematical practices, rather than as metaphysical absolutes, within the pragmatic domain.

What is most novel in this account is first, the identification of the foundations of epistemic certainty in purely symbolic derivations in the syntactical domain. This social constructionist account acknowledges there is certainty beyond what is guaranteed by the social rules and norms alone. This is a new and much needed response to critics of social constructionism (**Dieterle 2010)**. Second, this approach locates the epistemic values including the absolute certainty of mathematical results in the domain of axiology. It also finds the ontic values of the independence, existence and objectivity of mathematical objects in the domain of axiology. Although social constructionism repudiates the overall absolute certainty of mathematical knowledge, and the full metaphysical objectivity of mathematical objects, it accommodates both the beliefs in them, and the limited and partial realisation of these goods and values within mathematics. It is intended and hoped that these new analyses and concessions will be to the satisfaction of the working mathematician.

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