EXPLORING THE EDUCATIONAL POTENTIAL OF SYMMETRY IN SECONDARY MATHEMATICS

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ABSTRACT
Symmetry is a core concept that permeates the natural sciences, reflecting the essence of mathematical thinking through structural preservation and invariance under transformation. This study interprets symmetry not as an isolated topic or a collection of properties, but as a mathematical structure that organizes diverse mathematical content. Representative topics from the secondary mathematics curriculum—properties of quadrilaterals, similarity, cubic and quartic equations, and the normal distribution—were selected for theoretical analysis. The analysis shows that the properties and problem-solving methods associated with these topics are not incidental, but arise necessarily from shared structures of symmetry and invariance. Invariance under transformation organizes geometric properties in quadrilaterals, preserves ratios and relative positions in similarity, provides a basis for degree reduction in higher-order equations, and underlies standardization and the central limit theorem in the normal distribution. These results highlight symmetry as a mathematical structure that coherently connects mathematical concepts that may otherwise appear heterogeneous. By emphasizing transformation and invariance, this perspective offers a theoretical foundation for moving beyond memorization and procedural application toward structure-oriented understanding and reasoning in secondary mathematics education.

Keywords: higher-degree equations, normal distribution, properties of quadrilaterals, symmetry, similarity

1. Introduction

Although many teachers recognize that the various domains of mathematics are organically interconnected, classroom instruction in schools is often organized in a fragmented manner, with content presented separately by domain. If content from multiple mathematical areas can be integrated and explained around a unifying principle, students may experience a sense of coherence and structural harmony across what they learn. Such an approach may support learners in perceiving mathematics not as a collection of isolated topics, but as an interconnected body of knowledge.

This paper examine the meaning of symmetry, discuss how symmetry can function as a structural principle in mathematics, and explore ways in which the concept of symmetry can be integrated across different topics in secondary mathematics education.

Symmetry refers to the form or property of an object remain unchanged under certain transformations, and it constitutes a fundamental principle across mathematics and the natural sciences. In physics, symmetry has already served as a unifying paradigm to explain various laws comprehensively (Goldberg, 2013). According to Noether's first theorem, if the action of a physical system is invariant under a specific continuous transformation, the corresponding physical quantity is conserved. For instance, invariance of physical laws under temporal changes corresponds to the conservation of energy, invariance under spatial translations corresponds to the conservation of momentum, and invariance under rotations corresponds to the conservation of angular momentum. Moreover, in the theory of relativity, physical laws are considered consistent when the structure of spacetime is accounted for in all inertial frames.

In mathematics, symmetry also emerges as a central principle that determines structure, and the properties associated with symmetry play a central role. In linear algebra, real symmetric matrices are invariant under the transpose transformation that interchanges rows and columns. Owing to this symmetry, real symmetric matrices necessarily have real eigenvalues, eigenvectors corresponding to distinct eigenvalues are orthogonal, and such matrices can always be diagonalized by an orthogonal matrix. These essential properties follow inevitably from the underlying symmetry.

An isomorphism is a function that verifies the preservation of structure, and it appears in various forms, including algebraic isomorphisms, topological homeomorphisms, and differential diffeomorphisms. For example, a homeomorphism in topology is a function that compares two distinct topological spaces, and its bijectivity and continuity remain unchanged under inversion. This guarantees that two spaces can be regarded as identical through continuous deformations such as stretching or bending. The Riemann Hypothesis asserts that all nontrivial zeros of the Riemann zeta function in the complex plane have a real part equal to $\frac{1}{2}$. The invariance of the real parts, even within the complex distribution of prime numbers, suggests latent regularities underlying the primes. In this way, across various areas of advanced mathematics, symmetry functions as a foundational structure from which essential properties are generated.

However, in the context of mathematics education, symmetry is typically treated as content limited to particular units or as a property of specific geometric figures, and it is rarely employed systematically as a principle that organizes different mathematical domains. This tendency stems from curricula that present mathematical topics in a fragmented manner rather than emphasizing their structural connections. Organizing multiple mathematical ideas around a single theme or principle offers two major advantages. First, it can increase learners' interest by satisfying their desire to understand content in a coherent manner and to recognize causal relationships (Abbott, 2002). Second, experience in applying a single principle across diverse contexts enables learners to apply that principle more effectively in novel situations (Perkins & Salomon, 1992).

Accordingly, this study aims to interpret symmetry not as an isolated topic or property, but as a structuring principle that organizes diverse content within the secondary school mathematics curriculum. Specifically, the study analyzes the properties of quadrilaterals, similarity, higher-degree equations, and the normal distribution, and theoretically examines how the properties and problem-solving methods observed in each domain are generated from the shared mathematical structure of symmetry and invariance. This perspective allows seemingly disparate mathematical content to be viewed through a unified lens and suggests educational possibilities for supporting students in moving beyond memorization of individual properties or procedural application toward a structural understanding grounded in transformation and invariance.

2. Theoretical Background

This section clarifies the meaning of symmetry, reviews prior studies related to the theme of symmetry, and presents the research questions of this study. Mathematical knowledge beyond the secondary school curriculum that is necessary for understanding the main analysis is separately provided in the Appendix.

Symmetry as a Mathematical Structure

Historically, symmetry referred to “something with a balanced form” in ancient Greece (Weyl, 1952). Exemplars of symmetry included meticulously proportioned architecture, bilateral symmetry, regular polygons, and regular polyhedra. From the 19th century onwards, mathematicians such as Abel and Galois investigated the general solutions of quintic equations in relation to the symmetric properties of roots, leading to the development of group theory. Consequently, the notion of symmetry expanded to denote “invariance under transformation” (Stewart, 2007). Accordingly, traditional symmetric figures are described as remaining unchanged under transformations such as rotation and reflection, and symmetric polynomials are explained as being invariant under variable substitution.

In mathematics education, prior studies addressing symmetry have similarly emphasized its meaning as invariance under transformation. For instance, Nam & Park(2002) defined symmetry as “unchanged under certain transformations,” while Leikin et al.(2000) conceptualized symmetry as comprising three elements—objects (S), properties (Y), and transformations (M)—where an object belongs to the domain of transformations, and its properties remain unchanged when a transformation is applied. In the present study, symmetry is defined as “a form or property that remains unchanged when a transformation is applied,” reflecting its historical connection to form in ancient Greece and its relation to properties in modern mathematics and physics.

Symmetry Across Secondary Mathematics

Prior research on symmetry in mathematics education has predominantly explored two themes. The first concerns geometric and artistic symmetry—such as tessellations, fractals, and the golden ratio—highlighting aesthetic appeal (Weyl, 1952; Mandelbrot & Frame, 2002; Fathauer, 2020; Walser, 2024). The second investigates pedagogical approaches for teaching group theory and symmetry (Dubinsky et al., 1994; Burn, 1996; Larsen, 2013; Larsen & Lockwood, 2013; Gadanidis et al., 2018). Although closely related to symmetry, both approaches present educational limitations: the former offers limited integration with standard school curricula, and the latter, being highly abstract, may inadvertently foster misconceptions if applied rigidly (Freudenthal, 1973).

Other studies have examined more accessible topics for school mathematics, including “symmetry and equations” (Kiernan, 1971; Hamburg, 1976; Sheldon & Gordon, 1999; Ko & Lee, 2014; Kim et al., 2015), “problem solving through symmetry” (Dreyfus & Eisenberg, 2000; Leikin et al., 2000; Nam & Park, 2002), and “transformations and properties of figures” (Hong, 2005; Bonotto, 2007; Norton, 2019).

Research on “symmetry and equations” traced the historical efforts to solve higher-degree polynomial equations (Kiernan, 1971) and summarized Lagrange’s approach of employing symmetric polynomials for general solution methods (Hamburg, 1976; Ko & Lee, 2014), which has been applied in first-year high school mathematics classes (Kim et al., 2015).

Investigations also extended to analyzing the inflection points of cubic equations and exploring derivatives in quartic and higher-degree equations (Sheldon & Gordon, 1999). Research on "problem solving through symmetry" interpreted a variety of mathematical challenges from a symmetry perspective, such as determining the shortest path via point reflection (Leikin et al., 2000), expressing the golden ratio equation $x^2 - x - 1 = 0$ through continued fractions and infinite radicals, analyzing magic squares (Dreyfus & Eisenberg, 2000), proving the perpendicularity of cube diagonals and planes, verifying the uniqueness of the circumcenter of a triangle, and exploring formulas within Pascal's triangle as suggested by Freudenthal and Polya(Nam & Park, 2002).
The theme of "transformations and properties of figures" is strongly influenced by Klein's Erlangen Program, which sought to unify diverse geometries(Euclidean, affine, projective, hyperbolic, etc.) under the principle that geometry studies properties invariant under transformation. In mathematics education, this perspective shifts the focus from figures themselves to the operations that preserve their essential properties, providing students not only with logical proof skills but also with opportunities for active construction(Bonotto, 2007; Norton, 2019). Furthermore, invariants under affine transformations have been effectively employed to explain the centroids of polygons, properties of ellipses, and characteristics of parabolas in secondary mathematics, as well as to solve various construction problems(Hong, 2008).
However, much of this work was difficult to align with the secondary mathematics curriculum, lacked comprehensiveness across multiple mathematical domains, and did not present a coherent system as a mathematical structure.

Mathematical Structure

This study interprets symmetry as one of the important mathematical structures and focuses on understanding diverse mathematical knowledge in a systematic way. To this end, we specify the meaning of mathematical structure in order to establish an analytical perspective, and we identify its components so that they can be connected to educational implications. When mathematical structure is explicitly recognized, mathematical activity is no longer based solely on inductive experience; instead, it enables systematic prediction and purposeful action(Mason et al., 2009). Teachers also acknowledge that students need to understand mathematical structure. However, classroom observations indicate that teachers' instructional discourse reflects only a superficial understanding and use of mathematical structure(Gronow et al., 2022). In other words, although appreciating mathematical structure is considered important, it is not adequately addressed in school practice.
Appreciating mathematical structure does not mean possessing a large quantity of mathematical concepts; rather, it involves connecting concepts and grasping them as relations. However, structure requires additional conditions beyond relations alone. Mason et al. (2009) explain that the mathematical definition of a relation, derived from set theory, is a subset of a Cartesian product of sets. Under this definition, relations may be artificial or abstract and can include cases that carry little inherent meaning. For this reason, Mason describes structure as follows: "when a relationship is seen as an instantiation of a property, the relation becomes (part of) a structure." This study adopts this definition. In this study, symmetry is treated as a type of mathematical structure. It possesses a property such that, even when a transformation is applied to an object, its form or essential properties remain unchanged. Various topics in secondary mathematics are introduced as instantiations of this property.

To teach mathematical structure in a way that allows students to recognize it explicitly in classroom instruction, it is necessary to consider its components. Mason (2009) proposed five states: Holding wholes, Discerning details, Recognising relationships, Perceiving properties, and Reasoning on the basis of identified properties. Meanwhile, Gronow et al. (2022), in order to analyze teacher-directed communication, revised Mason's framework and proposed four components: connections to other learning, recognising patterns, identifying similarities and differences, and generalising and reasoning. These four components are not hierarchical and do not imply any fixed sequence.

3. Methods & Materials

This study aims to address a broader range of topics than previous research while restricting the content to knowledge that can be found within the secondary mathematics curriculum, thereby enhancing its practical usefulness in school settings. To examine whether curriculum content can be systematically explained through the properties of symmetry, four topics were selected: properties of quadrilaterals, similarity, higher-degree equations, and the normal distribution. Prior studies related to each topic were reviewed and used as supporting evidence for the analysis.

In deriving educational implications through analysis from the perspective of mathematical structure, this study also incorporates the components of mathematical structure proposed by Gronow et al. (2022). 'Connections to other learning' are reflected by explaining content across multiple mathematical domains in an integrated manner. 'Recognising patterns' is addressed by explicitly explaining how the properties of symmetry are established. 'Identifying similarities and differences' is reflected by distinguishing properties unique to symmetry (and absent in asymmetry). Finally, 'generalising and reasoning' is incorporated by clarifying what kinds of reasoning become possible through the use of symmetry. Based on these Methods & Materials, this study addresses the following research questions :

Research Question

(1) How does symmetry manifest in major content domains of secondary school mathematics (properties of quadrilaterals, similarity, higher-degree equations, and the normal distribution)? That is, with respect to which transformations is invariance established?

(2) What distinctive properties (those not present in asymmetric situations) are derived from this symmetry?

(3) What strategies for reasoning and problem solving can be provided through the use of symmetry?

4. Main Discussion

Symmetry in Quadrilaterals: Invariance under Geometric Transformations

In secondary mathematics, various types of quadrilaterals are often presented as collections of definitions and properties. However, these properties are not arbitrarily listed; rather, they can be systematically organized by a shared structure—namely, invariance under geometric transformations. In this study, the properties of quadrilaterals are interpreted not as isolated facts but as consequences of the actions permitted by geometric transformations. Quadrilaterals exhibit different levels of invariance with respect to geometric transformations such as rotations and reflections, and these differences in invariance are directly linked to differences in their geometric properties. Ellis-Davies(1986) systematically categorized the symmetry elements and corresponding properties of quadrilaterals, providing a theoretical basis for interpreting quadrilateral properties as outcomes of underlying symmetry structures.

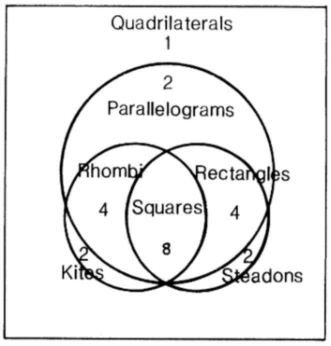
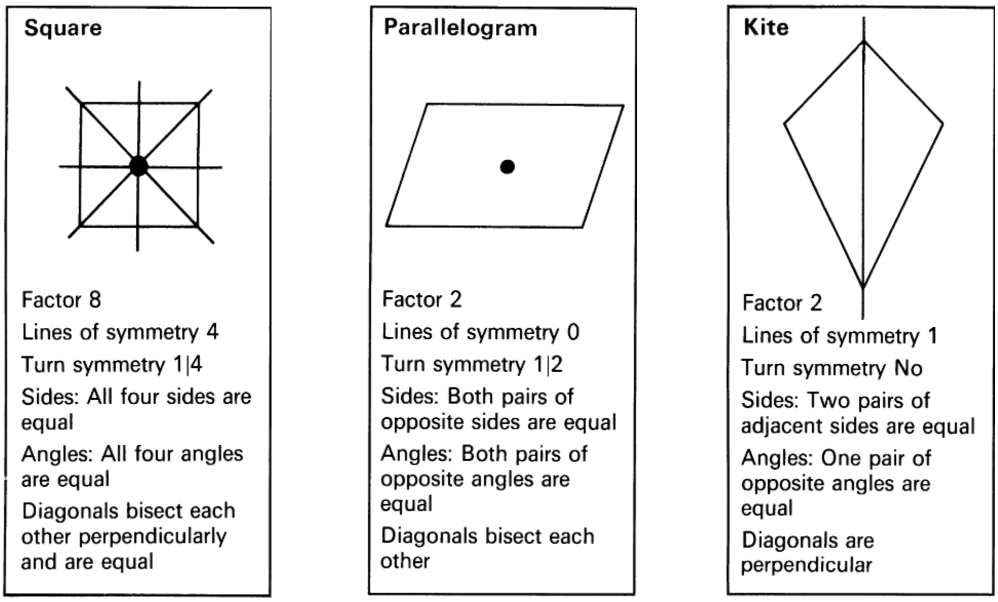


Figure 1. Inclusion Relationships among Quadrilaterals (Ellis-Davies, 1986)



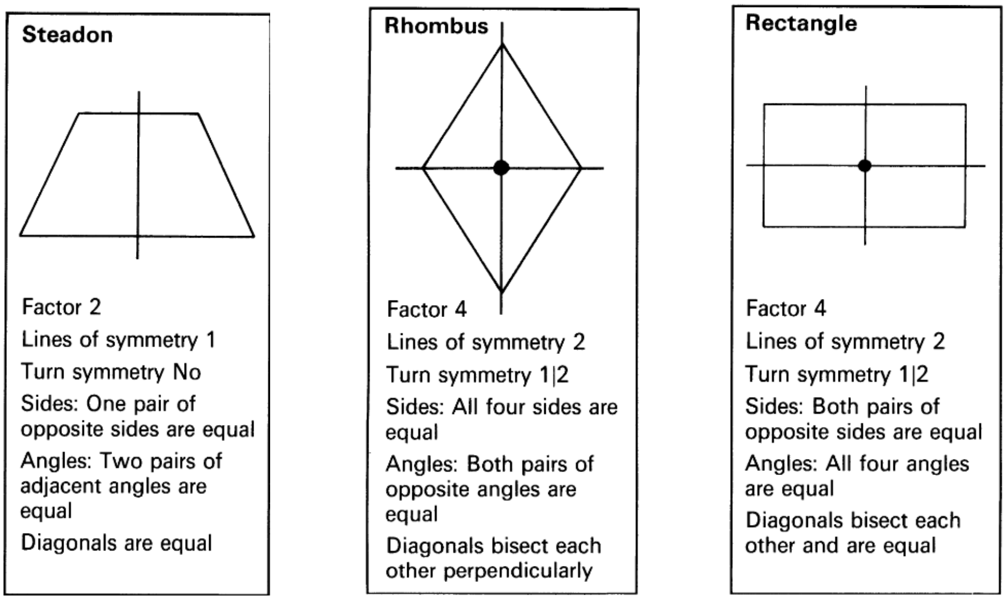


Figure 2. Factors and Properties of Various Types of Quadrilaterals (Ellis-Davies, 1986)

In this context, the symmetry factors of quadrilaterals dictate that the corresponding side lengths or angles become equal, and an increase in the number of symmetry factors generally enriches the properties of the figure. For example, a square is invariant under eight transformations—four rotations and four reflections. This rich symmetry structure necessarily entails a wide range of properties concerning side lengths, angle measures, and relationships between diagonals. In contrast, a parallelogram is invariant only under a 180° rotation. As a result, while properties related to corresponding sides and angles are preserved, additional properties arising from reflectional symmetry do not appear. From this perspective, the apparent "quantity" of properties associated with different quadrilaterals should not be understood as a matter of memorization load, but rather as a reflection of structural differences in the transformations each figure admits.[1]

Interpreting quadrilateral properties through the lens of symmetry has two important implications for students' mathematical thinking. First, when properties are understood as consequences of symmetry transformations, students move beyond observing the static appearance of fixed figures and toward reasoning about actions applied to mathematical objects and their resulting effects. This approach aligns with Piaget's characterization of mathematical development(Norton, 2019).
Second, problem-solving guided by symmetry can facilitate the discovery of efficient strategies. For instance, reflectional symmetry may inform the optimal placement of auxiliary lines, while rotational symmetry may aid in identifying corresponding angles. In this manner, lengths and angles can be determined through strategic reasoning rather than memorization of formulas or incidental discovery.[2]

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[1] Just as a group in group theory does not always inherit all properties of its subgroups, caution is required when interpreting symmetry structures solely through inclusion relationships among figures. In this discussion, symmetry is not employed as a schematic criterion for classifying quadrilaterals, but as a theoretical device for explaining the mechanisms through which geometric properties are generated.

[2] A discussion analogous to the properties of quadrilaterals can also be developed for triangles—for example, isosceles and equilateral triangles. The circle, moreover, represents the most symmetric planar figure: the

Symmetry in Similarity: Invariance of Ratio and Relative Position under Scaling

Similarity is commonly explained as a relationship in which the shape of a figure is preserved even when its size changes. However, such an explanation focuses solely on visual resemblance and fails to reveal a structural perspective that clarifies the meanings of related concepts, problem-solving strategies, and the axiomatic framework underlying similarity. This study reexamines the significance of ratios of lengths and relative positions that remain invariant under scaling transformations.

First, the preservation of length ratios under scaling transformations between similar figures provides a theoretical foundation for effectively determining unknown lengths in a wide range of geometric problems. Trigonometric ratios defined in right triangles are also situated within this same structure. The fact that the values of sine, cosine, and tangent remain constant across right triangles with the same angles is because trigonometric ratios are defined as quantities invariant under scaling transformations, rather than as measures dependent on the size of a particular triangle. This invariance forms the conceptual basis for the generalization of trigonometric ratios into trigonometric functions.

Moreover, trigonometric functions serve as effective tools in situations where unknown lengths must be determined. For example, trigonometric functions are employed in the formula for the area of a triangle $S = \frac{1}{2}ab\sin C$, the law of sines $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, and the law of cosines $a^2 = b^2 + c^2 - 2bc\cos A$, all of which rely on invariance under scaling to connect angles and lengths in a coherent structure.

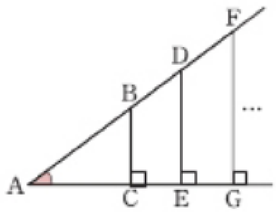

Figure 3. Similar Right Triangles

Ratio \ A	30°	45°	60°
sinA	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cosA	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tanA	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Table 1. Trigonometric Ratios of Special Angles

The symmetry structure of similarity preserves not only ratios of lengths but also the relative

perpendicular bisector of a chord invariably passes through the center(reflectional symmetry), while the correspondence between the length of an arc and the associated central and inscribed angles(rotational symmetry).

positions of points within a figure. The fact that special points—such as the center of a circle, the centroids of two similar figures, or the orthocenters of two similar triangles—occupy corresponding two points in similar figures that similarly situated demonstrates that similarity organizes invariance not merely of external shape but also of internal structure.[3]
This invariance of relative position plays a crucial role in Archimedes' study of the center of gravity. Archimedes accepted as an axiom that centers of gravity occupy corresponding relative positions under similarity, and on this basis he derived the centers of gravity of figures logically, without relying on empirical verification. In the proof presented below by Dijksterhuis (1987), all axioms and propositions employed rely on the invariance of relative position under similarity :

> Archimedes joins the supposed centre of gravity Θ to A, B, and Γ, and through the middle points E of AB and Z of AΓ draws straight lines parallel to AΘ, which meet BΘ and ΓΘ successively in K and Λ. In the similar triangles ABΓ and EBΔ, Θ and K are similarly situated points, therefore K is the centre of gravity of ΔEBΔ, and likewise Λ is the centre of gravity of ΔZΓΔ. Since further the areas of the triangles EBΔ and ZΓΔ are equal, the centre of gravity of their combination lies at the middle point of KΛ, i.e. the point of intersection N with ΘΔ. In the parallelogram AEΔZ, M is the centre of gravity; the centre of gravity of ΔABΓ therefore lies on MN, and cannot thus be Θ. Therefore Θ cannot lie outside AΔ.

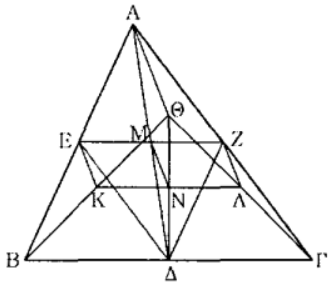

Figure 4. The Location of the Centroid of a Triangle

the proof relies on the principles that "similar figures have their centroids in same relative position"(Axiom 5), "if two figures of equal area are considered, the centroid of their union lies at the midpoint of the segment joining their individual centroids"(Proposition 4), and "the centroid of a parallelogram is located at the intersection of its diagonals"(Proposition 10) (Hong, 2005).
Interpreting similarity from this symmetry-based perspective has two important implications for students' mathematical thinking. First, students can construct problem-solving strategies by identifying ratios of lengths that are preserved under transformation. For example, when faced with a situation involving an unknown length, students may seek to obtain additional information about lengths or angles. Rather than memorizing formulas or algorithms, they can search for similar figures and develop strategies based on invariant similarity ratios or trigonometric ratios and functions. Second, instead of defining the centroid of a triangle merely as the intersection point of its three medians, deriving the centroid from axioms that

3 The statement "two points in similar figures are similarly situated" signifies that if one draws lines from an interior point of each figure to points lying on corresponding sides, the resulting angles are congruent(Dijksterhuis, 1987).

are naturally justified by symmetry allows students to experience proof not as the memorization of rules, but as a logical consequence grounded in the structure of similarity.

Symmetry in Higher-Degree Equations: Invariance under Permutations of Roots

The central principle in solving higher-degree equations of degree three or above does not lie in computational techniques such as synthetic division, but rather in identifying structures that remain invariant under substitutions of the roots. In secondary school mathematics, relatively simple cubic and quartic equations often admit integer solutions, allowing their degrees to be reduced mechanically. However, such approaches are no longer effective for general higher-degree equations. In these cases, the theoretical foundation that enables degree reduction lies in symmetric polynomials, which are invariant under permutations of the roots.
Let $x_1, \ldots, x_n$ denote the roots of an n-*th* degree equation. By the fundamental relationships between roots and coefficients, each coefficient is invariant under all permutations of the roots. That is, the elementary symmetric polynomials $x_1 + x_2 + \ldots + x_n$, $x_1x_2 + \ldots + x_{n-1}x_n$, ..., $x_1x_2\ldots x_n$ remain unchanged under the action of the full permutation group.

$$x^n - (x_1 + \ldots + x_n)x^{n-1} + (x_1x_2 + \ldots + x_{n-1}x_n)x^{n-2} - \ldots + (-1)^n x_1x_2\ldots x_n$$
$$= (x - x_1)\ldots(x - x_n)$$

This invariance guarantees that, although the individual values of the roots may be unknown, certain combinations of them are always determined once a higher-degree equation is given. Lagrange pursued a general method for solving higher-degree equations by further refining this invariant structure, an idea that later became foundational to Galois theory. Suppose that there exists a resolvent $\Phi(x_1, x_2, \ldots, x_n)$, a function of the roots, such that among all n! permutations of the roots, only two distinct values occur. In this case, the permutations split into two subsets of size $\frac{n!}{2}$, each yielding the same value, and each subset forms a subgroup of the full permutation group.
This group-theoretic structure provides a way to reorganize the set of roots into a simpler symmetric structure without directly computing the roots themselves. Following this idea, consider reducing an nnn-th degree equation to a quadratic problem. Suppose there exists a resolvent $\Phi(x_1, x_2, \ldots, x_n)$ that takes only two distinct values, denoted by Φ_1 *and* Φ_2, under all permutations of the roots. First, the value of $\Phi(x_1, x_2, \ldots, x_n)$ must be expressible as an algebraic combination of the elementary symmetric polynomials.[4] Second, it must be possible to reconstruct the original roots $x_1, \ldots, x_n$ from Φ_1 *and* Φ_2. In fact, the greatest difficulty in solving higher-degree equations lies precisely in finding a resolvent that satisfies this second condition. The key point is that $\Phi(x_1, x_2, \ldots, x_n)$ is not a fully symmetric function, but rather a function that exhibits symmetry only with respect to a specific subset of permutations.
Hamburg(1976) and Ko & Lee(2014) provide detailed accounts of how Lagrange employed resolvents in the solution of polynomial equations. For instance, in the case of a cubic equation with roots x_1, x_2, x_3, by the relationships between coefficients and roots: $x_1 + x_2 + x_3 = \sigma_1$, $x_1x_2 + x_2x_3 + x_3x_1 = \sigma_2$, $x_1x_2x_3 = \sigma_3$ are well established. If one constructs an appropriate resolvent function $\Phi(x_1, x_2, x_3) = (x_1 + wx_2 + w^2x_3)^3$(where $w^3 = 1, w \neq 1$), then among the $3! = 6$ permutations of the roots x_1, x_2, x_3, the values of Φ collapse into three pairs of identical values.

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[4] The first condition is guaranteed by the Fundamental Theorem of Symmetric Polynomials.

$\Phi_1 = \Phi(x_1, x_2, x_3) = \Phi(x_2, x_3, x_1) = \Phi(x_3, x_1, x_2)$
$\Phi_2 = \Phi(x_2, x_1, x_3) = \Phi(x_1, x_3, x_2) = \Phi(x_3, x_2, x_1)$

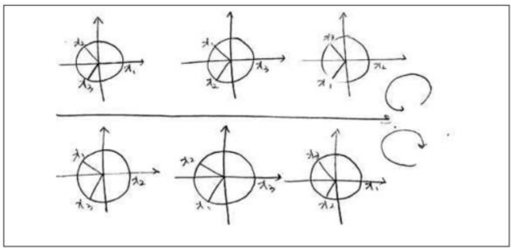

Figure 5. Representation of a Resolvent Polynomial $\Phi(x_1, x_2, x_3) = (x_1 + wx_2 + w^2x_3)^3$: Complex Plane (Kim et al.(2015))

In the case of distinct values Φ_1, Φ_2, the resolvent function can be expressed $\Phi_1 + \Phi_2 = 2\sigma_1{}^3 - 9\sigma_1\sigma_2 + 27\sigma_3$, $\Phi_1\Phi_2 = (\sigma_1{}^2 - 3\sigma_2)^3$ and by solving the corresponding quadratic equation, the values of Φ_1, Φ_2 can be obtained. Subsequently, by solving the following system of equations, all the roots of the original equation may be determined.

$$\begin{cases} x_1 + x_2 + x_3 = \sigma_1 \\ x_1 + wx_2 + w^2x_3 = \sqrt[3]{\Phi_1} \\ x_1 + w^2x_2 + wx_3 = \sqrt[3]{\Phi_2} \end{cases}$$

For a quartic equation, let the four roots be denoted as x_1, x_2, x_3, x_4. From the relations between coefficients and roots, the elementary symmetric polynomials $x_1 + x_2 + x_3 + x_4 = \sigma_1$, $x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = \sigma_2$, $x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = \sigma_3$, $x_1x_2x_3x_4 = \sigma_4$ are immediately known. If an appropriate resolvent $\Phi(x_1, x_2, x_3, x_4) = x_1x_2 + x_3x_4$ is constructed, then among the $4! = 24$ permutations of the roots x_1, x_2, x_3, x_4, the values of Φ collapse into three classes, each containing eight identical values.

$$\begin{aligned}
\Phi_1 &= \Phi(x_1, x_2, x_3, x_4) = \Phi(x_1, x_2, x_4, x_3) = \Phi(x_2, x_1, x_3, x_4) = \Phi(x_2, x_1, x_4, x_3) \\
&= \Phi(x_3, x_4, x_1, x_2) = \Phi(x_3, x_4, x_2, x_1) = \Phi(x_4, x_3, x_1, x_2) = \Phi(x_4, x_3, x_2, x_1) \\
\Phi_2 &= \Phi(x_1, x_3, x_2, x_4) = \Phi(x_1, x_3, x_4, x_2) = \Phi(x_3, x_1, x_2, x_4) = \Phi(x_3, x_1, x_4, x_2) \\
&= \Phi(x_2, x_4, x_1, x_3) = \Phi(x_2, x_4, x_3, x_1) = \Phi(x_4, x_2, x_1, x_3) = \Phi(x_4, x_2, x_3, x_1) \\
\Phi_3 &= \Phi(x_1, x_4, x_2, x_3) = \Phi(x_1, x_4, x_3, x_2) = \Phi(x_4, x_1, x_2, x_3) = \Phi(x_4, x_1, x_3, x_2) \\
&= \Phi(x_2, x_3, x_1, x_4) = \Phi(x_2, x_3, x_4, x_1) = \Phi(x_3, x_2, x_1, x_4) = \Phi(x_3, x_2, x_4, x_1)
\end{aligned}$$

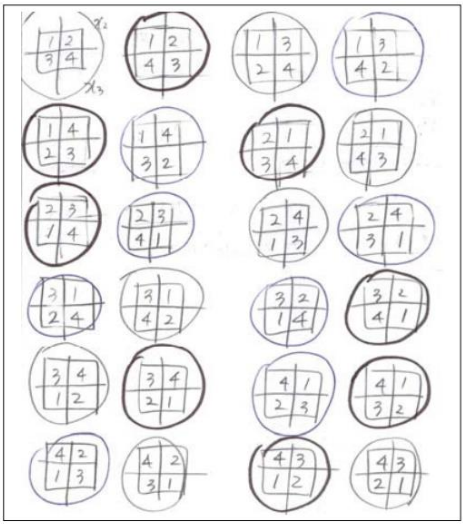

Figure 6. Representation of a Resolvent Polynomial $\Phi(x_1, x_2, x_3, x_4) = x_1x_2 + x_3x_4$ *: Diagonal Multiplication and Summation (Kim et al.(2015))*

With distinct values Φ_1, Φ_2, Φ_3, the resolvent function can be expressed $\Phi_1 + \Phi_2 + \Phi_3 = \sigma_2$, $\Phi_1\Phi_2 + \Phi_2\Phi_3 + \Phi_3\Phi_1 = \sigma_1\sigma_3 - 4\sigma_4$, $\Phi_1\Phi_2\Phi_3 = (\sigma_1{}^2 - 4\sigma_2)\sigma_4 + \sigma_3{}^2$ accordingly, and by solving the corresponding cubic equation, the values of Φ_1, Φ_2, Φ_3 can be obtained. Subsequently, by solving the following system of equations, all the roots of the original equation may be determined.

$$\begin{cases} x_1x_2 + x_3x_4 = \Phi_1 \\ x_1x_3 + x_2x_4 = \Phi_2 \\ x_1x_4 + x_2x_3 = \Phi_3 \\ x_1x_2x_3x_4 = \sigma_4 \end{cases}$$

It is worth noting, however, that the resolvent so obtained is not unique; indeed, Lagrange presented $\Phi(x_1, x_2, x_3, x_4) = x_1 - x_2 + x_3 - x_4$. For the quintic equation, however, the analogous method becomes overwhelmingly complex due to the combinatorial explosion of possible resolvents, and thus no general solution was ultimately found(Hamburg, 1976; Ko & Lee, 2014).

As discussed above, the general approach to solving higher-degree equations relies on exploiting elementary symmetric polynomials that are invariant under permutations of the roots. This approach can be understood as reconstructing the problem by extracting only the symmetric structure, while leaving the individual complexity of the roots intact. Beyond the elementary symmetric polynomials, additional symmetric polynomials—obtained by adding or multiplying resolvents $\Phi_1 + \Phi_2 + \Phi_3$, $\Phi_1\Phi_2 + \Phi_2\Phi_3 + \Phi_3\Phi_1$, $\Phi_1\Phi_2\Phi_3$—can also be constructed. By the fundamental theorem of symmetric polynomials, all such expressions can be represented as algebraic combinations of the elementary symmetric polynomials.

The role of symmetry becomes even more explicit in equations of special forms. For example, in equations of the form $x^6 + 2x^3 + 1 = 0$ where the degrees of all terms are multiples of n, the equation remains unchanged under the substitution x into ωx, ω^2x. This symmetry allows the equation to be reduced to a quadratic equation through the substitution $x^3 = t$. Similarly, equations of the form $x^4 + 4x^3 + 3x^2 + 4x + 1 = 0$, whose coefficients are symmetric from left to right, are invariant under the substitution x into $\frac{1}{x}$. Such equations can be reduced to quadratic equations by introducing the substitution $x + \frac{1}{x} = t$. From this perspective, variable substitution in equation solving should not be regarded as an arbitrary choice. Rather, substitutions should be selected by identifying transformations under which the equation remains invariant. In other words, effective problem-solving strategies emerge precisely when the structure of an equation admits symmetry.

Instruction grounded in this symmetry-based perspective also has important implications for students' mathematical thinking about higher-degree equations. Instead of habitually repeating computational procedures such as synthetic division, students can be encouraged to examine under which variable substitutions a given equation remains invariant. This awareness allows them to recognize that certain substitutions are not accidental but are necessitated by the symmetry of the equation, thereby enabling effective degree reduction. In the long term, such experiences may support students in identifying the symmetric structures underlying permutations of roots in general higher-degree equations, and can serve as a conceptual foundation for understanding group theory more meaningfully.

Symmetry in the Normal Distribution: Invariance under Linear Transformation

The normal distribution is one of the most widely used probability distributions for dealing with errors, formulating hypotheses about natural phenomena, and conducting regression analysis. In this study, however, the significance of the normal distribution is interpreted not from the perspective of statistical computation, but from the structural viewpoint of symmetry and invariance. While many probability distributions exist in statistics, the normal distribution occupies a distinctive position in that its form is preserved under specific transformations.

First, the normal distribution exhibits perfect bilateral symmetry around its mean, resulting in a bell-shaped curve. Owing to this symmetry, the mean, median, and mode coincide, providing a theoretical justification for using the mean as a representative measure of central tendency. Moreover, the distribution remains invariant under reflection about its mean, which allows for symmetric reasoning around the mean in the construction of confidence intervals and hypothesis testing.

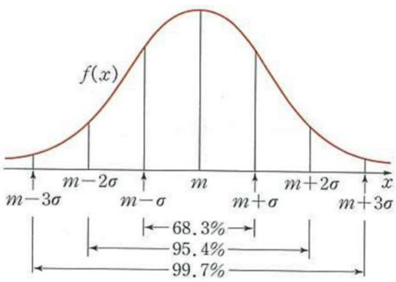

Figure 7. Confidence Intervals of the Normal Distribution

Second, the normal distribution is closed under linear transformations. That is, if random variables X, Y follow normal distributions, then linear combinations such as $aX + b$, $aX + bY$(where a, b are constants) also follow a normal distribution. Owing to this invariance, a random variable $X \sim N(m, \sigma^2)$ can be easily standardized as $Z = \frac{X-m}{\sigma}$, yielding $Z \sim N(0, 1^2)$. This property allows probabilities for all normal distributions to be computed using a single standard normal distribution table. Furthermore, linear combinations of independent random variables with diverse distributions gradually converge to a normal distribution. Thus, the

tendency for empirical data from many natural phenomena—often influenced by multiple factors—to approximate a normal distribution is not coincidental, but rather a necessary consequence that can be explained by invariance under linear combinations.
These structural characteristics of the normal distribution are made explicit in the central limit theorem. According to the theorem, regardless of the underlying population distribution, the distribution of the sample mean converges to a normal (Gaussian) distribution as the sample size increases. This is not merely an approximation result; rather, it reflects the fact that repeated linear transformations—such as summing independent random variables or taking their averages—gradually eliminate the fine details of the original population distribution, yielding a distribution that is invariant under such transformations. In this sense, the central limit theorem holds necessarily because it follows the symmetry structure permitted by the normal distribution.

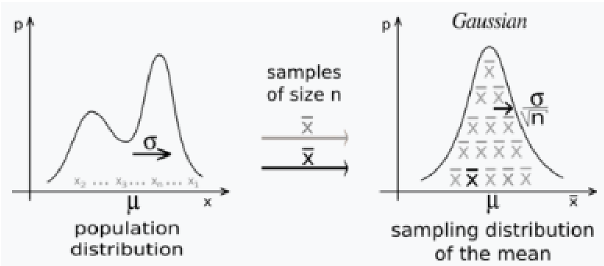

Figure 8. Central Limit Theorem

Teaching the normal distribution from the perspective of symmetry brings about two important shifts in students' mathematical thinking. First, students come to understand when it is reasonable to assume normality by focusing on invariance under linear combinations, rather than relying on procedural rules. Second, activities that express various normal distributions in terms of the standard normal distribution allow students to experience how diverse distributions can be reduced to a single underlying structure. As a result, the convenience of confidence interval calculations is perceived not as a procedural shortcut, but as a structural necessity grounded in symmetry.

5. Conclusion

This study examined the properties of quadrilaterals, similarity, higher-degree equations, and the normal distribution within the secondary school mathematics curriculum through the lens of symmetry, focusing on invariance under transformation. The analysis demonstrated that the various properties and solution methods presented in each domain are not the result of incidental discoveries, but rather are necessarily derived from structures that remain invariant under specific transformations. Whereas previous mathematics education research on symmetry has largely focused on localized topics or its application to particular problem situations, this study is distinctive in that it reconceptualizes symmetry as a structuring principle that organizes secondary mathematics content.

This perspective has the potential to transform students' mathematical thinking in secondary mathematics learning. When students move beyond memorizing individual properties or formulas and instead explore which transformations are relevant and what remains invariant under those transformations, mathematical objects can be perceived not as static outcomes but as dynamic constructs of reasoning. Such a way of thinking supports a coherent and structural understanding of mathematical knowledge and contributes to the inference of effective strategies for solving novel problem situations.

More specifically, the properties of quadrilaterals can be organized around invariance under geometric transformations; similarity highlights the significance of the preservation of length ratios and the relative positions of points under scaling transformations; in higher-degree equations, symmetry under permutations of roots provides the logical foundation for reducing the degree of equations; and in the normal distribution, invariance under linear transformations functions as a core principle underlying standardization and statistical inference. In this way, mathematical topics that may initially appear disparate can be structurally connected through the perspective of symmetry.

Of course, symmetry is not the sole means of explaining all concepts in secondary mathematics, and there exist important mathematical ideas that are inherently asymmetric. For example, unlike mathematical probability, statistical probability varies across trials, yet it is widely used to describe real-world phenomena. Even in such cases, however, investigating the conditions under which stability under transformation can be secured remains meaningful. Although statistical probability may initially lack reliability compared to mathematical probability, the law of large numbers demonstrates that a sufficiently high level of reliability can be achieved through repeated trials.

This study is limited to theoretical analysis and did not directly examine its effects within actual teaching and learning contexts. Future research should therefore develop instructional designs, teacher prompts, and learning tasks based on the perspective of symmetry proposed in this study, and empirically investigate their effects on students' conceptual understanding and reasoning.

Compared to asymmetric structures, symmetric ones tend to possess more intrinsic properties whose truth can be established through their invariance. Thus, symmetry is not merely aesthetically pleasing because it appears stable or orderly; its hidden properties carry profound value. In this sense, symmetry is indeed beautiful.

References

Abbott, H, P. (2002). *The Cambridge introduction to narrative*. Cambridge University Press.

Burn, B. (1996). What are the fundamental concepts of group theory?. *Educational Studies in Mathematics, 31*(4), 371-377.

Bonotto, C. (2007). The Erlangen Program Revisited: A Didactic Perspective. *For the Learning of Mathematics*, *27*(1), 33–38. http://www.jstor.org/stable/40248559

Dijksterhuis, E, J. (1987). *Archimedes*. New Jersey: Princeton university press.

Dubinsky, E. et al. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics, 27*(3), 267-305.

Dreyfus, T., & Eisenberg, T. (2000). On symmetry in school mathematics. *Visual Mathematics, 2*(5). http://eudml.org/doc/257086

Ellis-Davies, A. (1986). Symmetry in the Mathematics Curriculum. Mathematics in School, 15(3), 27–30. http://www.jstor.org/stable/30214083

Freudenthal, H. (1973). What groups mean in mathematics and what they should mean in mathematical education. In A.G. Howson(Ed.). *Proceedings of ICME-2: Developments in mathematical education*. Cambridge university press. 101-114.

Fathauer, R. (2020). *Tessellations: Mathematics, art, and recreation*. A K Peters/CRC Press.

Goldberg, D. (2013). *The Universe in the Rearview Mirror: How Hidden Symmetries Shape Reality.* Plume.

Gadanidis et al. (2018). Group Theory, Computational Thinking, and Young Mathematicians. *Mathematical Thinking and Learning, 20*(1), 32–53. https://doi.org/10.1080/10986065.2018.1403542

Gronow, M., Mulligan, J. & Cavanagh, M. (2022). Teachers' understanding and use of mathematical structure. *Math Ed Res J 34*, 215–240. https://doi.org/10.1007/s13394-020-00342-x

Hamburg, R.R. The theory of equations in the 18th century: The work of Joseph Lagrange. *Arch. Hist. Exact Sci. 16*, 17–36 (1976).

Hong, G, J. (2005). Misunderstandings and Logical Problems Related to the Centroid of a Polygon. *School Mathematics 7*(4), 391-402. https://koreascience.kr/article/JAKO200511722561666.pdf

Hong, G, J. (2008). Understanding the properties of geometric figures through the linear transformation and its implication for school mathematics. *The Mathematical Education 47*(4), 437-445. https://koreascience.kr/article/JAKO200804050607716.pdf

Kiernan, B.M. The development of Galois theory from Lagrange to Artin. *Arch. Rational Mech. 8*, 40–154 (1971).

Ko, Y, M. & Lee, S, W. (2014). Lagrange and Polynomial Equations. *Journal for History of Mathematics 27*(3), 165-182. http://dx.doi.org/10.14477/jhm.2014.27.3.165

Kim, J, H. et al. (2015). Teaching the Solutions of Equation in view of Symmetry. *Communications of Mathematical Education 29*(4), 699-722. https://www.kci.go.kr/kciportal/ci/sereArticleSearch/ciSereArtiView.kci?sereArticleSearchBean.artiId=ART002049110

Ko, Y, M. & Lee, S, W. (2021). On Symmetric Functions. *Journal for History of Mathematics 34*(2), 39-54. https://www.kci.go.kr/kciportal/ci/sereArticleSearch/ciSereArtiView.kci?sereArticleSearchBean.artiId=ART002716789

Larsen. S., & Lockwood. E. (2013). A local instructional theory for the guided reinvention of the quotient group concept. *The Journal of Mathematical Behavior, 32*(4), 726-742. https://doi.org/10.1016/j.jmathb.2013.02.010

Larsen. S. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior, 32*(4), 712-725. https://doi.org/10.1016/j.jmathb.2013.04.006
Mandelbrot, B., & Frame, M. (2002). *Fractals, graphics, and mathematics education*. Cambridge University Press.
Mason, J., Stephens, M. & Watson, A. (2009). Appreciating mathematical structure for all. *Math Ed Res J 21*, 10–32. https://doi.org/10.1007/BF03217543
Nam, J, Y., & Park, S, Y. (2002). Problem solving and teaching 'group concept' from the point of symmetry. *Journal of Educational Research in Mathematics 12*(4), 509-521. https://koreascience.kr/article/JAKO200211921475615.pdf
Norton, A. (2019). An Erlangen program that empowers students' mathematics. *For the Learning of Mathematics, 39*(3), 22–27. https://www.jstor.org/stable/26854430
Perkins, D. N., & Salomon, G. (1992). Transfer of learning. In T. Husén, & T. N. Postlethwaite (Eds.), *The international encyclopedia of education* (2nd ed., pp. 425-441). Oxford: Pergamon.
Stein, S. (1999). *Archimedes: What did he do beside cry eureka?*. Mathematical Association of America.
Sheldon, P., & Gordon (1999). On symmetries of polynomials, *PRIMUS, 9*(1), 13-20. https://doi.org/10.1080/10511979908965912
Stewart, I. (2007). *Why beauty is truth : A history of symmetry*. Basic Books.
Weyl, H. (1952). *Symmetry*. Princeton University Press.
Walser, H. (2024). *The golden ratio: Geometric and number theoretical considerations*. Springer.

APPENDIX

1. Mathematical Inquiry into the Centroid : Similarity

Archimedes sought to investigate the physical phenomenon of equilibrium (the centroid of figures) mathematically(Dijksterhuis, 1987). He adopted the most reliable empirical observations as axioms and then developed mathematical theory deductively(Hong, 2005). Some of the axioms and propositions he formulated are as follows:

Axiom 1: If two bodies of equal weight are placed at equal distances, they are in equilibrium. If placed at unequal distances, they are not in equilibrium, tilting toward the body placed farther away.
Axiom 2: If two bodies are in equilibrium, and an additional body with weight is added to one side, the balance tilts toward the added side.
Axiom 3: Similarly, if two bodies are in equilibrium and weight is removed from one side, the balance tilts toward the opposite side.
Axiom 4: If two figures are congruent, their centroids coincide.
Axiom 5: If two figures are similar but not congruent, their centroids are located at the same relative position.
Axiom 6: If certain magnitudes balance at a particular distance, then substituting equivalent magnitudes at the same distance preserves equilibrium.
Axiom 7: The centroid of a convex figure lies within the figure.

Proposition 4: The centroid of two figures of equal area lies at the midpoint of the segment joining

their individual centroids.
Proposition 10: The centroid of a parallelogram is the intersection point of its diagonals.

2. Lagrange's Equations: Higher-order Equations

Lagrange analyzed earlier mathematicians' solutions to cubic and quartic equations and identified their common underlying principle. His theorem later laid the foundation for group theory and became a core idea of Galois theory(Hamburg, 1976; Ko & Lee, 2021).

Symmetric polynomial: A multivariable polynomial invariant under permutation of variables.
Elementary symmetric polynomials: For n variables, these are $x_1 + x_2 + \dots + x_n$, $x_1x_2 + \dots + x_{n-1}x_n$, ..., $x_1x_2\dots x_n$
Resolvent(auxiliary equation): A function constructed from the roots of the original equation.
Solvable by radicals: A polynomial equation $x^n + a_1x^{n-1} + \dots + a_n = 0$ whose solutions can be expressed by an algebraic formula involving its coefficients through the four arithmetic operations and radicals.
Fundamental Theorem of Symmetric Polynomials: Every symmetric polynomial can be expressed as a polynomial in the elementary symmetric polynomials.
Lagrange's Theorem (Group Theory): For a finite group G, the order of any subgroup H divides the order of G.
Lagrange's Theorem 1: If a rational function $f(x_1, x_2, \dots, x_n)$ of n variables takes m distinct values under permutations of the variables, and if the number of permutations fixing a particular value is k, then $m = \frac{n!}{k}$.

3. Linear Combination: Normal Distribution

Linear combination: An expression formed by multiplying each term by a constant and summing the results.